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Dynamic scaling of coupled nonequilibrium interfaces

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We propose a simple discrete model to study the nonequilibrium fluctuations of two locally coupled (1+1)-dimensional systems (interfaces). Measuring numerically the tilt-dependent velocity we construct a set of stochastic continuum equations describing the fluctuations in the model. The scaling predicted by the equations is studied analytically using dynamic-renormalization-group theory and compared with simulation results.

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The dynamics of equilibrium and nonequilibrium interfaces has attracted much attention recently [1]. The continuum theory proposed by Kardar, Parisi, and Zhang (KPZ) [2] provided a very successful analytic approach for different problems, ranging from spin systems to growth models. However, usually the dynamics of interest is dominated by a strong-coupling fixed point, making the perturbative approach inconclusive. In such a situation simple discrete models allow us to obtain detailed information on the dynamic scaling.

In this paper a generalized single-step model is introduced to study the nonequilibrium dynamics of two coupled $(1+1)$-dimensional systems. Investigating the tilt dependence of the growth velocity we find that the set of nonlinear equations describing the fluctuations of the two surfaces $h_0(x,t)$ and $h_1(x,t)$ is

$$\partial_t h_0 = \nu_0 \partial_x^2 h_0 + \lambda_0 (\partial_x h_0)^2 + \gamma_0 \partial_x h_0 \partial_x h_1 + \varphi_0 (\partial_x h_1)^2 + \eta_0, \tag{1a}$$

$$\partial_t h_1 = \nu_1 \partial_x^2 h_1 + \lambda_1 (\partial_x h_1)^2 + \gamma_1 \partial_x h_0 \partial_x h_1 + \varphi_1 (\partial_x h_0)^2 + \eta_1, \tag{1b}$$

where ∂_t and ∂_x denote the partial derivatives with respect to t and x. The noise is assumed to have zero mean and uncorrelated $\langle \eta_i(x,t)\eta_i(x',t')\rangle = D_i \delta(t-t')\delta(x-x')$ with $i = 0,1$. The scaling of the fluctuations is determined numerically from the discrete model and the results are compared with the predictions of the dynamic-renormalization-group (DRG) calculations. A slightly modified version of the model allows us to study the evolution of a growing interface perturbed by a nonequilibrium field. The increase of the roughness exponent of the surface observed numerically is supported by the analytic predictions.

Equation (1) represents the most general form describing the fluctuations of two *coupled nonequilibrium* systems with *local* dynamics. It describes the evolution of two interfaces that are moving with the same mean velocity in an inhomogeneous media and are coupled by some local interaction. More generally it describes the growth of an interface $h_0(x,t)$ in the presence of a nonequilibrium field $h_1(x,t)$, assuming that the interface and the field are strongly affecting each other's behavior. In this respect it might provide a suitable analytic framework with which to describe the recent surface growth experiments in which the control of other physical factors was not possible. Such a perturbing field might be the density or pressure of the fluid during fluid displacement in porous media [3], the concentration of the nutrient in bacterial growth [4], or the density of the mesoscopic particles and impurities in the imbibition experiments [5].

On the other hand, the study of Eq. (1) is of general interest in the context of the recent efforts to understand the general properties of the nonequilibrium stochastic systems. Recently, Ertas and Kardar (EK), studying the motion of a single flux line in random environment, have presented a detailed investigation of (1) for $\gamma_0 = \lambda_1 = \varphi_1 = 0$ [6]. Imposing an additional condition, $\varphi_0 = 0$, the problem reduces to the convection of a passive scalar field $[T \equiv h_1(x,t)]$ in a random velocity field ($\mathbf{v} \equiv \nabla h_0$) [7]. For $\gamma_1 = \varphi_1 = 0$, Eq. (1) describes the fluctuations of an interface $h_0(x,t)$ evolving under the influence of an *independent* nonequilibrium field $h_1(x,t)$ [8].

The model. Our goal is to introduce a simple model to study the scaling of the two coupled interfaces. For this we use the mapping between the single-step model and the driven hard-core lattice gas on a one-dimensional chain [9]. In the mapping each step along the surface $h_0(x,t)$ is associated with a site on the chain H_0. A site j is said to be occupied [i.e., $H_0(j) = 1$] if we have an upward step on the interface $h_0(x=j,t)$, and is empty [$H_0(j) = 0$] otherwise. Two one-dimensional lattices, $H_0(j)$ and $H_1(j)$ with $(j = 1, L)$, are filled with probability $\frac{1}{2}$. First choose randomly a site j on one of the lattices (for example, H_0). If it is filled [i.e., $H_0(j) = 1$], the quantity $\vartheta = [H_0(j-1)+H_0(j)+H_0(j+1)-1.5][H_1(j-1) + H_1(j) + H_1(j+1) - 1.5]$ determines the direction in which the chosen particle wants to move. If $\vartheta > 0$ the $H_0(j)$ particle will try to move to the left, otherwise to the right. The step can be completed only if the neighboring site in the chosen direction is empty. Then choose randomly another particle on the other lattice and repeat the same procedure. The unit of time is defined as L trials per lattice, where L is the system size. Periodic boundary conditions are imposed on both of the lattices.

The quantity $[H_0(j-1) + H_0(j) + H_0(j+1)]$ is related to the local slope of the interface $h_0(x=j,t)$; thus with a positive ϑ the two interfaces have the same slope and generate a growth on site j, while a negative ϑ corresponds to different slopes and results in a decrease of the height.

We are interested in the scaling of the fluctuations characterized by the dynamic scaling of the width $w_i^2(L,t) = \langle [h_i(x,t) - \bar{h}_i(t)]^2 \rangle = L^{2\chi_i} f(t/L^{z_i})$, where χ_i is the roughness exponent for the interface $h_i(x,t)$, and the dynamic exponent z_i describes the scaling of the relaxation times with length; $\bar{h}(t)$ is the mean height of the interface at the moment t and the $\langle\,\rangle$ symbols denote ensemble and space average. The scaling function f_i has the properties $f_i(u \to 0) \sim u^{2\beta_i}$ and $f_i(u \to \infty) \sim$ const, where $\beta_i = z_i/\chi_i$.

Tilt dependence. In order to identify the relevant nonlinear terms determining the scaling behavior in the model we have measured the tilt-dependent velocity [10], imposing a global slope on the interfaces $h_0(x,t)$ and $h_1(x,t)$. A positive slope is induced on $h_0(x,t)$ by increasing the concentration of the H_0 particles; a negative slope is a result of a decrease in concentration. Figure 1 shows the tilt-dependent velocity v_0 and v_1, where $v_i = \sum_{j=1,L} \langle \partial_t h_i(x=j,t) \rangle$. The velocities which may explain the observed behavior and are compatible with the $x \to -x$ symmetry of the model have the form $v_0 = k - a_0(\partial_x h_0 \partial_x h_1) - b_0(\partial_x h_0)^2$ and $v_1 = k + a_1(\partial_x h_0 \partial_x h_1) - b_1(\partial_x h_0)^2$, where k, a_i, b_i are positive constants. The tilt in $h_0(x,t)$ induces a negative *mixing* term $(\partial_x h_0 \partial_x h_1)$ in v_0 and a positive one in v_1. But since in the model the influence of the $h_0(x,t)$ to $h_1(x,t)$ is identical to the influence of $h_1(x,t)$ to $h_0(x,t)$ (symmetric coupling), without tilt one expects the same sign for both mixing terms. The symmetric coupling is responsible for the presence of a $\varphi_0(\partial_x h_1)^2$ term in v_0 and a $\lambda_1(\partial_x h_1)^2$ term in v_1, which can be observed if the tilt is in $h_1(x,t)$ instead of $h_0(x,t)$. Based on these measurements we conclude that Eq. (1) contains all the relevant nonlinear terms determining the scaling of the two interfaces, and it describes the proposed model if $\lambda_i < 0$ and $\varphi_i < 0$. Higher-order terms might be responsible for the unusual behavior observed for large tilts, but they are in fact irrelevant concerning the scaling.

Dynamic-renormalization-group analysis. The scaling behavior of Eq. (1) in general can be investigated using one-loop DRG calculations [7, 11]. Rescaling the parameters $x \to e^{\ell}x$, $t \to e^{\ell z}t$, and $h_i \to e^{\ell \chi_i} h_i$, we obtain the following flow equations for the coefficients:

$$\frac{d\nu_0}{d\ell} = \nu_0\left[z - 2 + \frac{K_1}{\nu_0}\left(\frac{\lambda_0^2 D_0}{\nu_0^2} + \frac{\gamma_1\varphi_0 D_1}{2\nu_1^2}\right) + \frac{K_1\gamma_0}{2\nu_0(\nu_0+\nu_1)}\left(\frac{2\varphi_1 D_0}{\nu_0} + \frac{\gamma_0 D_1}{\nu_1}\right) + \frac{K_1\gamma_0}{\nu_0}\frac{\nu_0-\nu_1}{(\nu_0+\nu_1)^2}\left(\frac{2\varphi_1 D_0}{\nu_0} - \frac{\gamma_0 D_1}{\nu_1}\right)\right],$$

$$\frac{dD_0}{d\ell} = D_0\left[z - 2\chi_0 - 1 + \frac{\lambda_0^2 D_0 K_1}{\nu_0^3} + \frac{\gamma_0^2 D_1 K_1}{\nu_0\nu_1(\nu_0+\nu_1)}\right] + \frac{\varphi_0^2 D_1^2 K_1}{\nu_1^3},$$

$$\frac{d\lambda_0}{d\ell} = \lambda_0[z + \chi_0 - 2] + \frac{K_1}{\nu_0+\nu_1}\left(\frac{2\varphi_1 D_0}{\nu_0} - \frac{\gamma_0 D_1}{\nu_1}\right)\left(\frac{\gamma_0\lambda_0}{\nu_0}\frac{\nu_0-\nu_1}{\nu_0+\nu_1} + \frac{2\varphi_0\varphi_1}{\nu_1} - \frac{\gamma_0\gamma_1}{\nu_0+\nu_1}\right), \tag{2}$$

$$\frac{d\gamma_0}{d\ell} = \gamma_0[z + \chi_1 - 2] + \frac{K_1}{\nu_0+\nu_1}\left[\left(\frac{2\varphi_1 D_0}{\nu_0} - \frac{\gamma_0 D_1}{\nu_1}\right)\left(\gamma_0\frac{(\gamma_0 - 2\lambda_1)}{\nu_0+\nu_1} - \frac{\lambda_0\varphi_0}{\nu_0} + \frac{\gamma_1\varphi_0}{2\nu_1}\right) + \left(\frac{2\varphi_0 D_1}{\nu_1} - \frac{\gamma_1 D_0}{\nu_0}\right)\left(\gamma_0\frac{\gamma_1 - 2\lambda_0}{\nu_0+\nu_1} + \frac{\lambda_0\gamma_0}{2\nu_0} - \frac{\varphi_0\varphi_1}{\nu_1}\right)\right],$$

$$\frac{d\varphi_0}{d\ell} = \varphi_0[2\chi_1 - \chi_0 + z - 2] + \frac{K_1}{\nu_0+\nu_1}\left(\frac{\gamma_1 D_0}{\nu_0} - \frac{2\varphi_0 D_1}{\nu_1}\right)\left(\varphi_0\frac{\gamma_1 - 2\lambda_0}{\nu_0} + \gamma_0\frac{\gamma_0 - 2\lambda_1}{\nu_0+\nu_1}\right).$$

The flow equations for the other five missing coefficients can be obtained from (2), replacing $\nu_0 \leftrightarrow \nu_1$, $D_0 \leftrightarrow D_1$, $\lambda_0 \leftrightarrow \lambda_1$, $\gamma_0 \leftrightarrow \gamma_1$, $\varphi_0 \leftrightarrow \varphi_1$, and $\chi_0 \leftrightarrow \chi_1$.

Obtaining the exponents from (2) is not straightforward because of the large number of parameters involved, but important results can be obtained by making use of the nonperturbative properties of (1) combined with the direct integration of (2).

If $2\varphi_0\nu_0 D_1 = \gamma_1\nu_1 D_0$ and $2\varphi_1\nu_1 D_0 = \gamma_0\nu_0 D_1$ [fluctuation-dissipation (FD) subspace] the joint probability [6]

$$\wp[h_0(x,t), h_1(x,t)] = \exp\left[-\int dx\left(\frac{\nu_0}{2D_0}(\partial_x h_0)^2 + \frac{\nu_1}{2D_1}(\partial_x h_1)^2\right)\right]$$

is a solution of the Fokker-Planck equation following from (1). This provides us the exact exponents $\chi_i = \frac{1}{2}$. In this case the nonlinear terms do not renormalize, resulting in the scaling relation $z + \chi_i = 2$. Direct integration of (2)

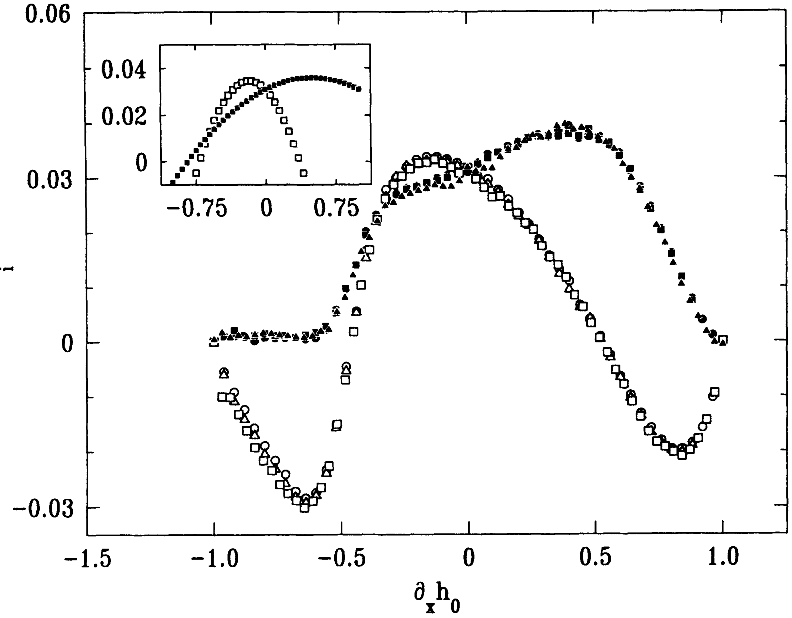

FIG. 1. The tilt-dependent velocity v_0 (empty symbols) and v_1 (filled symbols) for system sizes $L = 125$ (circles), $L = 250$ (triangles), and $L = 500$ (squares). Each set of data points was obtained by averaging over 250, 100, and 25 independent runs. The inset shows the simplest polynomial in the tilt $(\partial_x h_0)$ of the form $v_0 = 0.031 - 0.34(\partial_x h_0) - 0.12(\partial_x h_0)^2$ and $v_1 = 0.031 + 0.02(\partial_x h_0) - 0.02(\partial_x h_0)^2$, which may account for the observed behavior in the vicinity of zero. Higher-order terms may be responsible for the complicated behavior observed for large tilt.

shows that for positive α_i, φ_i, and γ_i the flow converges to the FD subspace, thus resulting in the superdiffusive exponents $\chi_i = \frac{1}{2}$ and $z = \frac{3}{2}$. A change of variables $h_0 \to -h_0$ and $h_1 \to -h_1$ indicates the same exponents if *all* the coefficients of the nonlinear terms are negative.

As the velocity measurements indicated, $\lambda_i < 0$ and $\varphi_i < 0$ in the model. If γ_i is also negative, the FD subspace dominates the behavior. The DRG does not provide exact results in the $\gamma_i > 0$ case; direct integration of (2) indicates a strong-coupling fixed point, with diverging λ_i, γ_i, and φ_i. The exponents in the model were determined using saturated systems of size $L = 50, 100, 200, 400$ and unsaturated systems of size $L = 1000, 2000, 4000$. The roughness exponent χ_i was obtained from the best collapse for the time-dependent width. All the simulations indicated superdiffusive exponents, giving as a result $\chi_i = 0.52 \pm 0.03$ and $\beta_i = 0.32 \pm 0.02$. Concluding this section we note that the simulations are in perfect agreement with the predictions of the DRG for $\gamma_i < 0$, so it is very likely that this is the sign of the mixing term in the model. But since no analytical results are available for $\gamma_i > 0$, we cannot rule out the possibility that for this sign the fluctuations of the two systems are also characterized by the superdiffusive exponents.

The $\lambda_0 = \varphi_0 = \varphi_1 = \gamma_1 = 0$ *case.* The h_1 interface fluctuates independently of h_0 in the following version of the model: The randomly chosen particle on the H_0 chain will try to move in the direction determined by the sign of ϑ, but the one chosen on H_1 will try to move left with a probability p and right with probability $1-p$, independently of ϑ. On H_1 we have the single-step model with

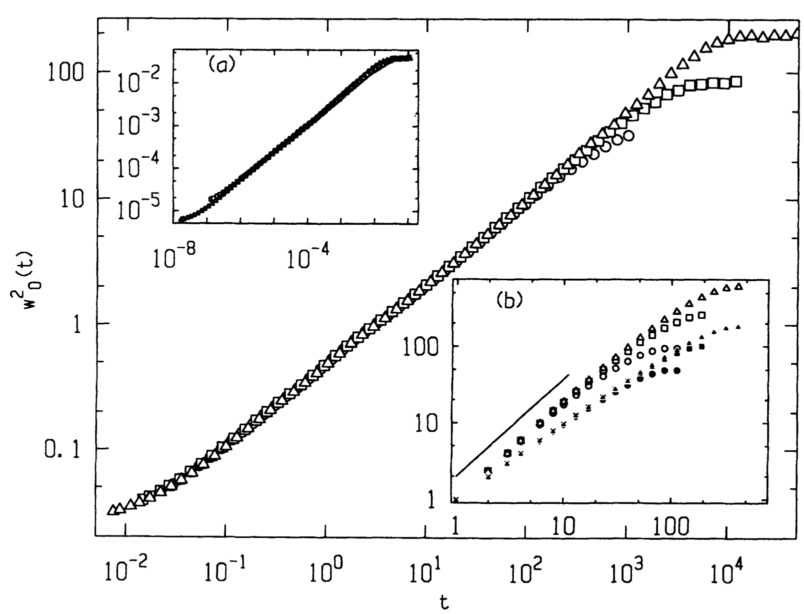

FIG. 2. The $\lambda_0 = \varphi_i = \gamma_1 = 0$ case: The scaling of the width with time for systems of size $L = 200$, 400, and 800. An average over 1000 independent runs was taken. The main figure shows the scaling after an intrinsic width [12] of magnitude 0.225 was extracted from the data. The slope of the straight part gives $\beta_0 = \beta_1 = 0.33 \pm 0.02$. The roughness exponent χ was determined from the best collapse of the data, shown in inset (a). Inset (b) shows the scaling of the height-height correlation function $\langle |h_i(x,t) - h_i(x+l,t)|^2 \rangle \sim l^{2\chi_i}$ for the interfaces $h_0(x,t)$ (empty symbols) and $h_1(x,t)$ (filled symbols), for the same system sizes as in the main picture. The slope of the straight line is $2\chi_0 = 1.28$.

evaporation. If $p \neq \frac{1}{2}$, the fluctuations of the h_2 interface are described by the KPZ equation; thus $\gamma_1 = \varphi_1 = 0$ in (1). For $p = 1$, measurements on the tilt dependence of v_0 indicate the absence of the λ_0 and φ_0 terms. But these results do not indicate the absence of the γ_0 term: the fact that it has no influence on the velocity may come from the vanishing contribution of $\langle \partial_x h_0 \partial_x h_1 \rangle$, which is acceptable, considering that h_1 fluctuates independently of h_0. The exponents determined from the model are $\chi_0 = 0.64 \pm 0.03$ and $\beta_0 = 0.33 \pm 0.02$, together with the known exponents of the KPZ equation: $\chi_1 = \frac{1}{2}$ and $\beta_1 = \frac{1}{3}$ (see Fig. 2). These exponents are in good agreement with those obtained by EK from the direct integration of Eqs. (1) for $\lambda_1 < 0$ and $\gamma_0 > 0$ [13]. The flow equations (2) show that λ_1 scales to zero; thus the DRG is not conclusive for these coefficients. It indicates only that $z_0 > z_1$, which agrees with the numerical findings. A support for the positivity of γ_0 comes from the DRG result for $\gamma_0 < 0$, the one-loop exponents being $\chi_0 = \frac{3}{4}$ and $\beta_0 = \frac{1}{2}$, considerably larger than those observed numerically in the model [14].

The $\lambda_0 = \lambda_1 = \varphi_0 = \varphi_1 = \gamma_1 = 0$ *case.* If in the second variant of the model we choose $p = \frac{1}{2}$, the h_1 interface fluctuates in equilibrium with $\lambda_1 = \varphi_1 = \gamma_1$, leading to the Edwards-Wilkinson (EW) exponents $\chi_1 = \frac{1}{2}$ and $\beta_1 = \frac{1}{4}$ [15]. The situation in the model is the following: the h_1 interface fluctuating in equilibrium influences the fluctuation of the h_0 interface through the mixing term $(\partial_x h_0 \partial_x h_1)$. The velocity measurements again confirm the absence of the φ_i, λ_i terms. The simulations indicate the nontrivial exponents $\chi_0 = 0.68 \pm 0.02$ and $\beta_0 = 0.34 \pm 0.02$, together with the EW exponents for h_1. The DRG in the present form cannot be applied because of the different scaling of the time in the two interfaces (i.e., $z_0 \neq z_1$).

In this paper we have focused mostly on those parameter values that were accessible through the studied discrete models. For symmetric coupling the model allowed us to determine the scaling exponents that were consistent with the exact results obtained from the DRG. The study of two particular cases was also possible using a modified version of the model. Both cases lead to considerable enhancement of the roughness exponent χ_0.

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