

Dynamics of Random Networks: Connectivity and First Order Phase Transitions

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Abstract

The connectivity of individual neurons of large neural networks determine both the steady state activity of the network and its answer to external stimulus. Highly diluted random networks have zero activity. We show that increasing the network connectivity the activity changes discontinuously from zero to a finite value as a critical value in the connectivity is reached. Theoretical arguments and extensive numerical simulations indicate that the origin of this discontinuity in the activity of random networks is a first order phase transition from an inactive to an active state as the connectivity of the network is increased.

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Networks of neuron type threshold elements have generated a lot of interest lately, motivated by their potential for reproducing neurobiological processes and understanding the generic mechanism governing basic brain functions. Most of the studied models deal either with fully connected networks (for which every neuron is connected to all neurons in the system [1]) or highly diluted networks [2,3]. At this point little is known about the properties of networks with arbitrary connectivity, including networks whose connectivity is realistic from the standpoint of biology.

Here we take a first step towards understanding the dynamical properties of random networks by investigating the effect of connectivity on the network dynamics [4], where the connectivity is the probability that two randomly selected neurons have a synaptic connection. A network with small connectivity is inactive, i.e. if we excite it with an external stimulus, the activity dies out in a short time. One would expect that increasing the connectivity leads to a gradual increase in the activity of the network. Contrary to this simple picture, here we show that upon increasing the connectivity, the steady state activity does not increase continuously, but jumps discontinuously, the network going through a *first order phase transition* from an inactive to an active state. This result may help us understand the evolutionary driving forces that lead to the synaptic densities characterizing the brain by providing the range of connectivities necessary for the network to display a nonzero activity, and sheds light into the difficulties occurring during modeling of low level sustained activity in the cerebral cortex [5].

Consider a network formed by N neurons. Each neuron is connected randomly to any of the other neurons with a probability ρ , i.e. ρ is measure of the network connectivity. The detailed wiring diagram of the system is given by the matrix \mathbf{J} , whose element J_{ij} is 1, if there is a directed path from neuron i to j , and zero otherwise.

First we consider a simplified version of the McCulloch-Pitts model [6], limiting the model to only excitatory neurons. Later we will show that this simplification does not affect the main conclusions of the paper. Every neuron i can be in two possible states, active ($s_i = 1$) or inactive ($s_i = 0$). Using synchronous updating, the state of neuron

i at time $t + 1$ is $s_i(t + 1) = 1$ with probability $P(h_i, T) = [1 + \tanh(h_i(t)/T)]/2$, and $s_i(t + 1) = 0$ otherwise. Here $h_i(t) = \sum_j J_{ji}s_j(t) - \theta$, θ is the threshold of the individual neuron, and T (temperature) characterizes the spontaneous firing of a neuron. The *activity* of the network can be characterized by the normalized number of active neurons at time t : $x(t) = (1/N) \sum_i s_i(t)$. Hereafter we call a network *active (inactive)* if $x(t) \neq 0$ ($x(t) = 0$), for large t .

After activating a fraction x of all neurons, if the network is highly diluted ($\rho \rightarrow 0$), the neurons are not connected to sufficient number of other neurons to overcome θ , and the activity decays to zero. In contrast, a highly connected network ($\rho \rightarrow 1$) for almost any initial conditions leads to an active state. The steady state dynamics in the active state is rather complex, leading to periodic oscillations at $T = 0$, whose period depends on N , ρ and \mathbf{J} . However, at nonzero temperatures the dynamics is quasiperiodic, i.e. the system explores randomly a number of periodic orbits.

Our goal is to understand the changes in the network activity as the connectivity ρ is varied. For this we implement a numerical method to calculate the *free energy*, that characterizes the dynamics of networks with fixed (N, ρ) . For now we limit ourselves to symmetric networks, for which $J_{ij} = J_{ji}$.

Consider the above network model after it reached its steady state and define $f'(x)$ as

$$x(t + 1) \equiv x(t) + f'(x). \quad (1)$$

Here $f'(x)$ depends on \mathbf{J} and on the initial conditions $\{x_i(t = 0)\}$. Averaging over the various realizations of the network topology, \mathbf{J} , leads to $f(x) = [f'(x)]_{J, x(0)}$, which is a smooth and univalued function. Alternatively, we can define the continuum version of (1), using the averaged $f(x)$

$$\partial_t x(t) = f(x(t)) + \eta(t), \quad (2)$$

where we added the uncorrelated noise, $\eta(t)$, to incorporate the randomness of the updating rule at nonzero temperatures.

Formally, (2) can be generated from the *free energy* $\mathcal{F}(x)$ using

$$\partial_t x(t) = -\frac{\partial \mathcal{F}(x)}{\partial x} + \eta(t), \quad (3)$$

where

$$\mathcal{F}(x) = -\int_0^x f(x') dx'. \quad (4)$$

As we show below, using (1) and (4), $\mathcal{F}(x)$ can be *calculated* using a mean field approximation and *measured* in numerical simulations.

Mean field theory (MFT)— At time t we randomly activate a fraction $x(t)$ of neurons. The goal is to calculate $x(t+1)$. The calculation proceeds in three steps.

(i) Choosing randomly a neuron, the probability $P(N, k, \rho)$ that it is connected to k other neurons follows the binomial distribution $P(N, k, \rho) = \binom{N-1}{k} \rho^k (1-\rho)^{N-1-k}$, where $\binom{a}{b} = a!/b!(a-b)!$.

(ii) Knowing that a fraction x of all neurons are active, the probability $\Pi(k, m, x)$, that of the k neurons the chosen neuron is connected to, m are active, is given by the binomial distribution $\Pi(k, m, x) = \binom{k}{m} x^m (1-x)^{k-m}$. (iii) Finally, a neuron with m active neighbors is activated with probability $P(m, T)$.

Thus, the activity of the network, $x(t+1)$, is given by

$$x(t+1) = \sum_{m=0}^{N-1} P(m, T) \sum_{k=m}^{N-1} P(N, k, \rho) \Pi(k, m, x(t)). \quad (5)$$

This expression provides $f(x) \equiv x(t+1) - x(t)$, and using (4), we can calculate the free energy $\mathcal{F}(x)$.

Obtaining (5) we made two approximations. In (i) we neglected the quenched nature of the randomness in the connectivity matrix \mathbf{J} , replacing \mathbf{J} with a new one at every time step, subject to fixed ρ and N . In (ii) we assumed that every neuron has the same probability x to be active, independent of the network topology. In reality the activity of a neuron is highly correlated with its connectivity. However, even within these approximations, the MFT captures correctly the nature of the phase transition from the inactive to the active state.

Phase transition— (a) The free energies obtained from (5) for $T = 0$, $N = 100$, $\theta = 2$, and various values of the connectivity ρ are shown in Fig. 1(a). For small ρ the only stable state has zero activity ($x = 0$), i.e. starting with any x , the activity decays. Increasing ρ , at $\rho_1(0) \approx 0.042$ the free energy develops an inflection point, leading to a second, local and unstable minima at $x_1(0)$, corresponding to a nonzero activity. Further increasing ρ , at $\rho_c(0) \approx 0.046$, $\mathcal{F}(x_1)$ becomes smaller than $\mathcal{F}(x = 0)$, indicating that the system undergoes a *first order phase transition* during which the activity jumps from $x = 0$ to $x_1 \neq 0$. The formally stable $x = 0$ state becomes a metastable state.

(b) For nonzero T , thermal fluctuations induce a thermal activity, $x_0(T) = [1 + \tanh(-\theta/T)]/2$, that is independent of the network topology. Expanding (5) for small ρ and x , we find that the derivative of the free energy in $x = 0$ is negative for nonzero T , thus \mathcal{F} has no minima at $x = 0$, but only at the thermally activated $x_0(T)$. For small T , increasing ρ leads to the appearance of the metastable $x_1(T) > x_0(T)$, which becomes a global minima at $\rho_c(T)$ (see Fig. 1(b)). Thus for small T we observe a first order phase transition from the thermally activated state, $x_0(T)$, to the active state $x_1(T)$.

(c) For high T the thermally induced activity, $x_0(T)$, dominates the dynamics of the system, such that for $T \geq T_c$ the local minima corresponding to the active state $x_1(T)$ does not appear, and we can not distinguish between the thermally activated state and the active state of the network (see Fig. 1(c)).

We summarize the above behavior using the (ρ, T) phase diagram, shown in Fig. 1(d). For small T the active and the thermally activated states are separated by the line $\rho_c(T)$. However, the $\rho_c(T)$ line ends at T_c , and for $T > T_c$ the system is too noisy to distinguish the thermally activated state from the true activity of the network.

Steady state behavior— Due to the discussed approximations, the MFT does not provide us the steady state behavior of the network. For this we measured the steady state $f(x)$ using the discussed network model. Fig. 2 shows the measured free energy for $T = 0$ and for $T > T_c$. For $T = 0$ we observe a first order phase transition from the inactive to the active states, as predicted by the MFT (see Fig. 2(a)). For $T \neq 0$, thermal fluctuations lead

to a thermally induced metastable state [7], which is indistinguishable from the active state of the network if $T > T_c$. Thus the phase diagram for the steady state behavior is similar to the one predicted by the MFT, shown in Fig. 1(d).

Both the MFT and simulations in the steady state predict the existence of a first order phase transition for small temperatures as the connectivity of the network is increased. A number of important questions arise at this point [8]: How generic is the observed first order phase transition? What are the implications of this transition?

To answer the first question, we studied extensions of the described model, including elements that are often considered in neural network simulations. Simulations show that introducing asymmetry ($J_{ij} \neq J_{ji}$) modifies the $\rho_c(T)$ curve, but the system still undergoes a first order phase transition. We find that the phase transition is not affected by the inclusion of inhibitory neurons either, nor by the nature of updating process (random or synchronous). Thus we conclude that the existence of a first order phase transition for small temperatures is a generic property of random networks, and it is related to the topology of the network rather than the dynamics of the individual neurons.

What determines the actual connectivity and the synaptic density of certain parts of the brain? Naturally, a complete answer to this question should consider the genetically determined non-randomness in the synaptic connections. However, our results suggest that for random networks the optimal connectivity depends on the dynamical properties desired for the particular network. When a certain fraction of neurons are activated by an external stimulus, the network should respond with nonzero activity. For most brain functions this activity should decay after some time if the stimulus is not sustained. For a large network this can be achieved by exciting the network into a metastable state. In this case the activity decays after some characteristic time, which depends on the free energy barrier between the stable and the metastable states. On the other hand, if sustained activity is the goal, one needs to use a ρ larger than $\rho_c(T)$. For a long time cerebral cortex modeling was halted by the inability of large random network models to reproduce the observed low level activities [5]. The observed first order phase transition explains the origins of this failure: the arbitrary

small activity between zero and $x(\rho_c(T))$ is simply not available in network models due to the jump in the activity at the transition point.

Furthermore, the outlined method (1-4) could be used in exploring the properties of large networks when only a small fraction of the neurons can be monitored in the laboratory. Examples include multi-electrode recording techniques, that provide information about the activity of typically tens of neural cells [9]. Such measurements may help understand the network activity by providing $\mathcal{F}(x)$ for small x .

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FIGURES

FIG. 1. The free energy predicted by the mean-field theory (5). (a) $T = 0$. The curves, from top to bottom, correspond to $\rho = 0.020, 0.042, 0.046, 0.050$. (b) $T = 1.11$. The curves, from top to bottom, correspond to $\rho = 0.020, 0.042, 0.043, 0.045$. (c) $T = 10$. The curves, from top to bottom, correspond to $\rho = 0, 0.020, 0.070, 0.101$. (d) The (T, ρ) phase diagram. The solid line corresponds to $\rho_c(T)$, separating the thermally activated phase from the active state. The error bars are smaller than the symbols.

FIG. 2. The steady state free energy obtained from numerical simulations for $N = 100$ and $\theta = 2$. Average over 1000 runs is taken. (a) $T = 0$. The curves, from top to bottom correspond to $\rho = 0.005, 0.010, 0.011, 0.015$. (b) $T = 10.0$. The curves, from top to bottom, correspond to $\rho = 0.0, 0.030, 0.043, 0.101$.



