# SUPERTRACKS AND nTH ORDER WINDOWS IN THE CHAOTIC REGIME

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The purpose of this paper is to generalize the concept of supertrack functions (STF), to sketch the main lines of a renormalization theory for STF and to obtain a scaling relation yielding $n$th order windows in the chaotic domain for a large class of one-dimensional maps.

The concepts of supertracks (ST) and supertrack functions (STF) were introduced by Oblow in a very interesting paper [1]. They refer to orbits starting from the maximum point of a one-dimensional map and proved to be quite useful in investigating the chaotic behaviour of, for instance, the quadratic map $F(\lambda, x)=\lambda x(1-x)$. It is however to be noted that ST and STF can be found, under different names, also in other works (e.g. refs. [2,3]) which use them to explain boundary shadings or, alternatively, peaks of probability distributions, in plots of chaotic attractors.

We found that the definition of the Oblow supertrack functions can be extended to a larger class of one-dimensional maps and that they can also be used to calculate the position of a subset of the periodic windows of an arbitrary order inside the chaotic regime. This can be obtained by means of scaling relations resulting from a renormalization scheme, the main lines of which we present in the remaining part of this paper.

We consider maps of the interval of the same type as those in refs. [4] and [5]. We write them in the general form

$$x_{n+1}=F(\lambda, x_n)=\lambda f(x_n) \tag{1}$$

and remind the essential properties of functions $f(x)$: (a) $f(x)$ is $C^0$, single-valued, piecewise $C^1$ on $[0,1]$ and strictly positive on $(0,1)$, $f(0)=f(1)=0$; (b) $f(x)$ has either a unique maximum or attains its maximum on a subinterval of $(0,1)$; (c) whenever $f(x)=f_{\max}$, $f'(x)$ exists and is equal to zero; (d) a $\lambda_0$ exists such that for $\lambda_0<\lambda_{\max}=1/f_{\max}$, $F(\lambda, x)$ has only two fixed points, of which one is the origin, and both are repellent.

For functions satisfying conditions (a) to (d), it was shown in ref. [4] that there exists an infinity of periodic windows. It would be thus useful to have a simple method giving the position of at least some of these windows. We will show in the following how this problem can be solved with the help of STF.

Among the various orbits of one dimensional maps, of particular interest are the superstable ones. These are orbits including the maximum point of the map, $F(x^*)$.

The $n$th order STF is defined by means of $F^n(\lambda, x^*)$, the $n$th iterate of $x^*$ as function of $\lambda$, as

$$s_n(\lambda)=F^n(\lambda, x^*) . \tag{2}$$

For the quadratic map, $F(x,\lambda)=\lambda x(1-x)$, this leads to the Oblow expression

$$s_{n+1}(\lambda)=\lambda s_n(\lambda)[1-s_n(\lambda)] .$$

For general maps, the first three STF are

$$s_0(\lambda)=x^* , \quad s_1(\lambda)=\lambda f(x^*)=\lambda f_{\max} ,$$
$$s_2(\lambda)=\lambda f(\lambda f_{\max}) . \tag{3}$$

The STF have some remarkable properties:

(1) The intersection of $s_n(\lambda)$ with $s_0(\lambda)$ gives the position $\lambda_n$ of the superstable point of the $n$th order periodic window as solution of the equation

$$s_n(\lambda_n)=s_0(\lambda_n) .$$

(2) By means of eq. (2) we get a simple relation between STF of different orders:

$$s_{n+1}(\lambda)=F(\lambda, s_n(\lambda)) . \tag{4}$$

(3) STF satisfy

$$s_n(\lambda_{\max})=0 \quad \text{for } n\geqslant 2 , \tag{5}$$

a relation which is easy to obtain taking into account that

$$s_1(\lambda_{\max})=1 , \quad s_2(\lambda_{\max})=\lambda_{\max}f(1)=0 .$$

(4) The general expression of the derivative of a STF at $\lambda_{\max}$ is

$$(\mathrm{d}s_{n+k}/\mathrm{d}\lambda)_{\lambda_{\max}}=(f_1\,\lambda_{\max})^k(\mathrm{d}s_n/\mathrm{d}\lambda)$$
$$=(f_1/f_{\max})^k(\mathrm{d}s_n/\mathrm{d}\lambda)_{\lambda_{\max}} \tag{6}$$

for $n\geqslant 2$ and with $f_1\equiv(\mathrm{d}f(x)/\mathrm{d}x)_{x=0}$.

STF or order $n$ have a very complicated structure (see our fig. 1 and figs. 2 and 3 in ref. [1]). However, the whole picture still retains a character of universality in that their general appearance is similar near the point $\lambda_{\max}$. To see this (fig. 2) we can choose, for instance, an $s_n\geqslant 2$ and notice first that each such $s_n(\lambda)$ has a last part, between $\lambda_{n-2}$ and $\lambda_{\max}$ where it increases from $s_2(\lambda_{n-2})$ to $s_1(\lambda_{n-1})$ falling down afterwards to $s_n(\lambda_{\max})$, i.e. to zero. If we now iterate $s_n(\lambda)$ to get $s_{n+1}(\lambda)$, we notice that:

(a) at $\lambda_{n-1}$, $s_{n+1}(\lambda)$ and $s_2(\lambda)$ have the same value;

(b) at $\lambda_n$, the intersection point of $s_n(\lambda)$ and $s_0(\lambda)$, $s_{n+1}(\lambda)$ is maximum;

(c) at $\lambda_{\max}$, $s_{n+1}(\lambda_{\max})=0$.

It appears therefore that $s_{n+1}(\lambda)$ has the same shape for $\lambda\in(\lambda_{n-1}, \lambda_{\max})$ as has $s_n(\lambda)$ for $\lambda\in(\lambda_{n-2}, \lambda_{\max})$, but slightly compressed, with the compression factor $\Lambda_{n+1}$ given by

$$\Lambda_{n+1}=\frac{\lambda_{\max}-\lambda_{n-2}}{\lambda_{\max}-\lambda_{n-1}} . \tag{7}$$

We can express this in a more compact way by means of a shift $\beta$ with respect to $\lambda_{\max}$:

$$s_n(\lambda_{\max}-\beta)\simeq s_{n+1}(\lambda_{\max}-\beta/\Lambda_{n+1}) ,$$

After $k$ iterations we thus obtain

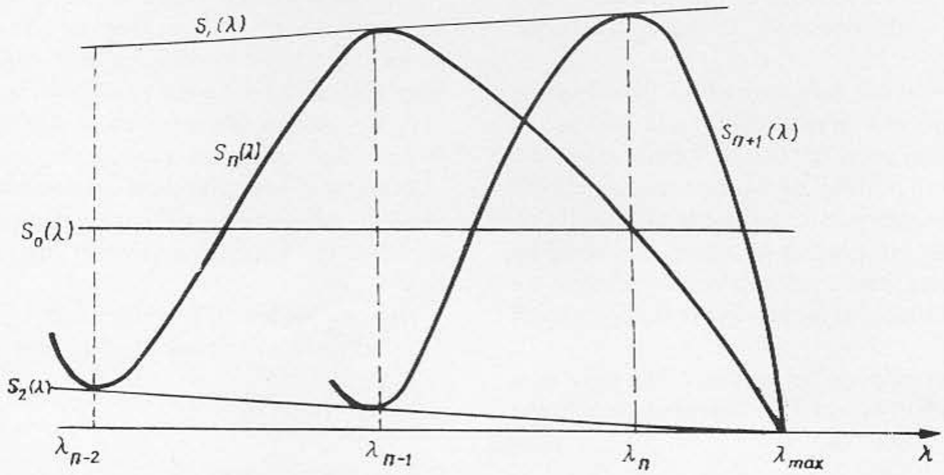


Fig. 1. The 8th order supertracks for the logistic map compared to the supertracks of orders 0, 1 and 2.

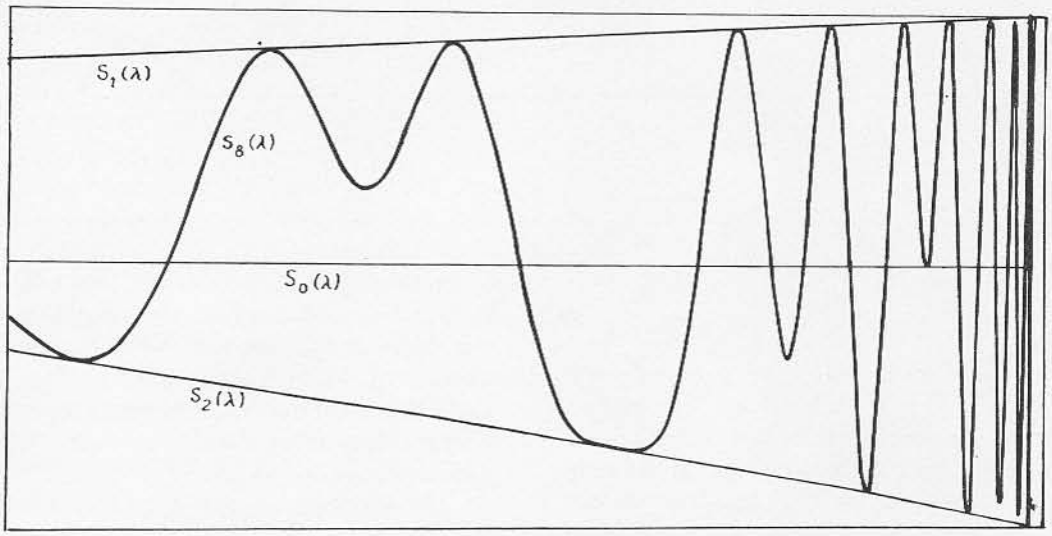


Fig. 2. The behaviour of supertracks of orders $n$ and $n+1$ near $\lambda_{\max}$.

$$s_n(\lambda_{\max}-\beta)\simeq s_{n+k}\left(\lambda_{\max}-\beta\left(\prod_{j=1}^{k}\Lambda_{n+j}\right)^{-1}\right). \quad (8)$$

One notices a slight difference in the values of the maxima but this cancels in the limit of large $n$ due to the rapid convergence of $\lambda_n$ to $\lambda_{\max}$. This scaling can be described as a two-step process leading to the definition of a general function $B$. We start from $s_n(\lambda_{\max}-\beta)$ and then (i) form

$$s_{n+1}(\lambda_{\max}-\beta)=(\lambda_{\max}-\beta)f(s_n(\lambda_{\max}-\beta)),$$

(ii) rescale

$$s_{n+1}(\lambda_{\max}-\beta)\to s_{n+1}(\lambda_{\max}-\beta/\Lambda_{n+1}), \quad |\Lambda|>1.$$

Let us denote by $B_n$ the set of operations (i) and (ii),

$$s_{n+1}(\lambda_{\max}-\beta)=B_n[s_n(\lambda_{\max}-\beta)].$$

We notice that (i) is $n$-independent and (ii) is asymptotically $n$-independent.

Numerical results indicate that the sequence $\{\Lambda_n\}$ converges to a limit $\Lambda$ and, consequently $B_n\to B$. One can now see that

$$\lim B^r[s_n(\lambda_{\max}-\beta)]=s^*(\lambda_{\max}-\beta),$$

where $s^*(\lambda_{\max}-\beta)$ is the solution of the functional equation

$$s^*=B(s^*).$$

$B(s^*)$ is thus a general function since it is independent of $n$, but one cannot consider it as universal due to its dependence on the function $f$ itself.

To calculate $\Lambda$ we Taylor expand $s_n(\lambda)$ and $s_{n+k}(\lambda)$ near $\lambda_{\max}$:

$$s_n(\lambda_{\max}-\beta)=\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\beta^j\left(\frac{d^js_n(\lambda)}{d\lambda^j}\right)_{\lambda_{\max}}, \quad (9)$$

$$s_{n+k}(\lambda_{\max}-\beta/\Lambda_{n+k})=\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\left(\frac{\beta}{\Lambda^*_{n+k}}\right)^j\left(\frac{d^js_{n+k}(\lambda)}{d\lambda^j}\right)_{\lambda_{\max}}, \quad (10)$$

$$\Lambda^*_{n+k}=\prod_{i=1}^{k}\Lambda_{n+i}.$$

Now, using relations (8)–(10), keeping only linear terms and by means of eq. (6), one obtains $(f_1/f_{\max})^k/\Lambda^*_{n+k}=1$.

For large $n$, one can write $\Lambda_{n+1}=\Lambda_{n+2}=...=\Lambda$, so that $\Lambda=f_1/f_{\max}$.

In this way, one can write eq. (6) in the form

$$(ds_n(\lambda)/d\lambda)_{\lambda_{\max}}=\Lambda^{n-1}f_{\max}. \quad (11)$$

Table 1

| Function | $f_{max}$ | $f_1$ | $\Lambda = f_1/f_{max}$ | Computed | Order of the periodic window |
|---|---|---|---|---|---|
| $Q(x)$ | 1/4 | 1 | 4 | 4.001 | 22 |
| $S(x)$ | 1 | $\pi$ | $\pi$ | 3.141 | 13 |
| $C(x)$ | 83/64 | 9 | 9.128... | 9.2 | 7 |
| $L(x)$ | 1 | 1/e | 2.22 | 2.3 | 7 |

One also can approximate the value of $\lambda_n$ by

$$\lambda_n \sim \Lambda_{max} - \Lambda^{-n} . \quad (12)$$

The value of $\Lambda$ computed for the same test functions as in ref. [4] is given in table 1. To compute $\Lambda$ we have always considered the last of the periodic windows of the order specified in the last column. (One can see that the larger the order of the periodic window, the better the precision of the computed value.)

One easily notices that the last $n$th order window corresponds to a U-sequence (cf. ref. [4]) $P_n = RL^{n-2}$. Calculating $\Lambda_n$ for the logistic curve for windows of the type $P_n = RL^{n-3}R$ corresponding to the intersection of the last increasing part of $s_n(\lambda)$ with $s_0(\lambda)$ shows that it converges to the same $\Lambda$ as described in eq. (11).

An important observation one can make is related to the fact that $\lambda_{max}$ corresponds to a crisis point. Therefore, one could say that whenever a crisis is approached, scaling relations of the same type as above should hold, with possibly different values of $\Lambda$. It would be thus of interest to calculate the factors affecting $\Lambda$ in such cases, which we intend to do in a further paper.

## References

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