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LETTER TO THE EDITOR

A model for temporal fluctuations of the surface width: a stochastic one-dimensional map

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Abstract. A stochastic one-dimensional map is introduced to model the steady-state fluctuations of the surface width in far-from-equilibrium surface roughening. The dynamics of the map and the correlations in the time sequence are investigated. In particular, for power law distributed noise a non-trivial multi-affine behaviour is observed.

The ubiquity of self-affinity in nature has attracted much interest recently. Considerable progress has been made in understanding the dynamics of non-equilibrium surface growth in the context of a variety of models, analytical theories and experiments [1, 2].

Kardar, Parisi and Zhang (KPZ) proposed [3] a nonliner Langevin equation which successfully explained many aspects of surface dynamics observed in different models. The equation describes the temporal development of the height variable $h(x, t)$:

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x, t) \tag{1}$$

where x is a d-dimensional position vector and η is a stochastic driving noise, whose correlation is specified by the two-point correlation function

$$\langle \eta(x, t)\eta(x', t')\rangle = 2D\delta(x - x')\delta(t - t').$$

Traditionally the noise is assumed to have a Gaussian distribution.

In (1) the first term on the right-hand side represents the relaxation of the surface by a surface tension ν, while the second term is attributed to the lateral growth.

In the absence of noise equation (1) describes the smoothing of a surface: given any rough initial configuration the surface will gradually approach a flat shape. However, the combined effect of independent noise and the nonlinear interaction in (1) makes the interface rough.

A central problem to be understood is the non-trivial coupling between noise and deterministic dynamics associated with different models.

The relevant physical observable is the average width $w(L, t)$, where L is the linear size of the system. It is expected to have the scaling form [1] $w(L, t) \sim L^{\alpha} f(t/L^{\alpha/\beta})$, where L is the system size and $f(u)$ has the asymptotic behaviour $f(u \to \infty) = \text{constant}$, $f(u \to 0) \sim u^{\beta}$.

The roughness can be characterized by the exponent α. For two-dimensional systems α and β are known to be exactly [3, 4]: $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$. Recent experiments on bacterial colony growth [5] and immiscible displacement of viscous fluids in porous media [6, 7] have led to exponents different from these: the results for α are in the range 0.75–0.85.

A possible explanation for the anomalously large exponents found in these experiments has been proposed by Zhang [8]. He suggested that the new exponents arise from the fact that the amplitude of the random noise in the experiments has a non-Gaussian, power law distribution of the form $P_n(\eta) \sim 1/\eta^{1+\mu}$ where η is the delta correlated noice. The μ dependence of the exponents α and β has been observed in numerical simulations both in two and three dimensions [9–11].

Recently numerical evidence has been presented of multi-affine scaling in a model of rare events dominated roughening [12]. For multi-affine surfaces the qth-order height–height correlation functions should be studied [13, 14]

$$c_q(x) = \langle |h(x') - h(x'+x)|^q \rangle_{x'} \sim x^{qH_q} \tag{2}$$

where H_q is an exponent continuously varying with q. In [12] a crossover length has been identified below which the qth-order correlation functions showed multiscaling behaviour. Although the existence of this crossover length made the situation similar to the effect of the intrinsic width on the scaling [15], the variation of the crossover length with the system size showed that the situation here is qualitatively different. This multi-affine scaling is specific to rare-events-dominated roughening; the Gaussian noise in the KPZ equation results in a constant H_q.

Extensive simulations have already been carried out to study the temporal development of correlations. The fluctuations of the width are less studied. In the steady state $t \gg L^{\alpha/\beta}$ the width $w(L, t)$ saturates and fluctuates around a constant which scales as L^α. In a recent paper[16] has been presented numerical and analytical results showing $1/f^\omega$ characteristic frequency spectra of the time series of surface width of a ballistic deposition model. This result can be understood if we consider that the time series generate a self-affine signal.

In this paper a stochastic one-dimensional map is presented in order to model some aspects of the temporal fluctuations in growth models. The model is very sensitive to the noise distribution: for bounded noise the time sequence is found to be self-affine while for power law distributed noise it shows multi-affine behaviour up to a critical timescale.

As was pointed out in [3] the determinstic version of the KPZ equation (i.e. with $\eta(x, t) = 0$), describes a smoothing phenomena. In two dimensions the solution in the asymptotic regime is composed of paraboloid segments

$$h_t(x) = A - \frac{(x-\xi)^2}{2\lambda t}. \tag{3}$$

This relation is the solution of (1) in the absence of the $\nabla^2 h$ term.

Considering such a paraboloid segment in moment t_0 with characteristic width L and height $h(L, t_0)$, its time evolution can be seen on figure 1. The effect of the deterministic KPZ equation on this paraboloid segment is the decrease of its height from $h(L, t_0)$ to $h(L, t+t_0)$:

$$h(L, t+t_0) = \frac{h(L, t_0)}{kh(L, t_0)+1} \tag{4}$$

where $k = 8\lambda t/L^2$ and $h(L, t_0) = A$.

Since for a paraboloid segment the width $w(r, t)$ is related linearly to the height $h(k, t)$, the height fluctuations of the paraboloids correspond to the width fluctuations of the surface.

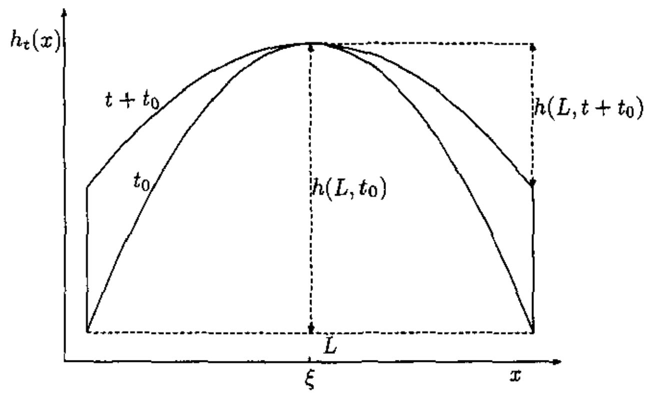

Figure 1. The evolution of a paraboloid segment in time: The paraboloid's height decreases from $h(L, t_0)$ to $h(L, t+t_0)$ under the affect of the smoothing described by (3).

The effect of the noise can be considered as increasing the height of the paraboloid by η. In this picture the fluctuations of the paraboloid's height are the effect of two different factors: the noise increases the height of the paraboloid from $h_t(=h(L, t_0))$ to $h_t+\eta$ and the deterministic dynamics decreases its height to h^{t+1}. This coupled phenomenon is described by a one-dimensional map mixed with noise

$$h_{t+1}=f_\eta(h_t) \qquad f_\eta(h_t)=\frac{h_t+\eta}{k(h_t+\eta)+1}. \tag{5}$$

At this point we should like to show the difference between shifting the paraboloid segments and varying their height. The former is incorporated in the constant A in (3), which can take values specified by the initial conditions $h(L, t_0)$ and it has no effect on the width. The latter is affected by the noise and the smoothing described by (4) and it is the quantity of interest. It is defined as $h(L, t)=h_t(\xi)-h_t(\xi+L/2)$ (see figure 1).

In the remainder of this letter we shall study the properties of the map (5) and the effect of k and η on the iteration.

For $\eta=0$ (no noise) the map (5) has a stable fixed point in zero, so every non-zero initial condition converges rapidly to this point. This corresponds to the smoothing phenomenon described by the deterministic KPZ equation: the roughness decreases in time leading to a flat surface. For non-zero η (we assume that the noise is positive, i.e. $\eta\geqslant 0$) the map has a non-zero fixed point

$$h^*(\eta)=\left(\frac{\eta^2}{4}+\frac{\eta}{k}\right)^{1/2}-\frac{\eta}{2}. \tag{6}$$

This fixed point is stable so during the iteration h_t approaches it. But η is a random variable so $h^*(\eta)$ appears as a fictive fixed point, which changes its position randomly. In the $k\to 0$ limit for non-zero η the expectation value of $h^*(\eta)$ scales with k as $h^*\sim k^{-1/2}(\sim L)$. Since the derivative of $f_\eta(x)$ in h^* is $f'_\eta(h^*)\to 1+O(k)$, in the $k\to 0$ limit this relation predicts a linear convergence to the fixed point. These results can be condensed into a simple scaling law similar to that used in surface growth [2]:

$$h(L, t)\sim L^\alpha\left(\frac{t}{L^{\alpha/\beta}}\right) \tag{7}$$

where the scaling function $g(u \to 0) \sim u^\beta$ and $g(u \to \infty) \approx$ constant. The scaling exponents are $\alpha = 1$ and $\beta = 1$, so the scaling relation $\alpha + \alpha/\beta = 2$ is fulfilled.

These exponents are independent of the noise distribution. Figure 2 presents numerical evidence for this scaling: the inset shows the time evolution of h_t for different k values; the main figure shows these curves rescaled following the scaling relation (7).

If $t \gg k^{-\alpha/\beta}$ the height h_t saturates and fluctuates around a constant value which scales as $k^{-1/2}$. We continue with investigating the correlations in the time dependent signal registered after the saturation. We study the scaling of the qth-order height–height correlation function applied to time dependent signals:

$$c_q(t) = \langle |h_{t+t_0} - h_{t_0}|^q \rangle_{t_0}$$

which is expected to scale as $c_q(t) \sim t^{qH_q}$. If H_q is constant the signal is simply self-affine, while for q-dependent H_q the signal is called multi-affine. In both cases H_1 plays the role of the roughness exponent [13].

If η has a Gaussian distribution, $c_q(t)$ is found to exhibit a simple scaling law with constant $H_q = 0.51 \pm 0.02$. The measurements on very long signals show that the fluctuations in h_t generate a self-affine function. We note that this value is the same as for the corresponding width fluctuations [16].

Non-trivial scaling has been observed when the noise η had a power law distribution. The scaling of the qth-order height–height correlation functions for different values of q can be seen on figure 3. The one million length signal used in this analysis was generated for the parameter values $k = 10^{-10}$ and $\mu = 3$. The initial part of the data set shows a multi-affine scaling with a scaling exponent H_q depending on q. A crossover to a self-affine scaling is observed if t exceeds some value t_c. A third region, for $t > t_{sat}$ is present, where the correlation functions saturate. This is the most relevant part of our results. We expect that multi-affine scaling might appear in the steady-state fluctuations of the width in rare-events-dominated roughening. A similar scaling has been observed when the spatial correlations of a such a surface has been investigated [12]. We note that for larger values of k the multi-affine scaling is also present but the second self-affine part cannot be observed clearly.

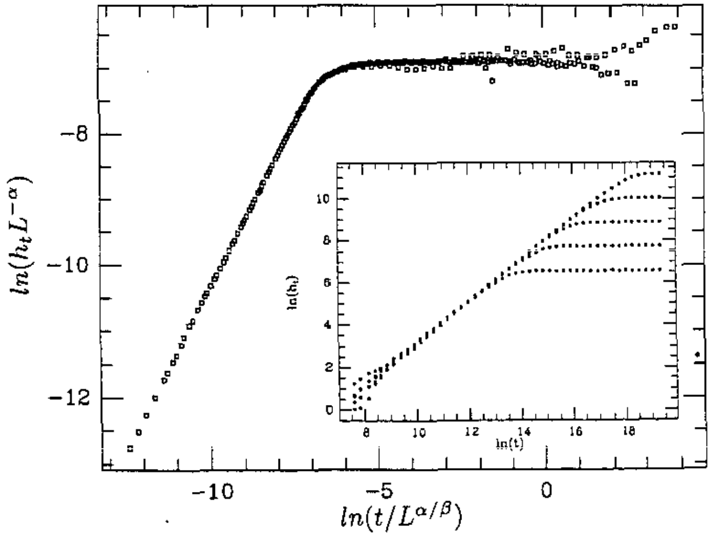

Figure 2. Time evolution of h_t for $k = 10^{-7}$, 10^{-8}, 10^{-9}, 10^{-10}, 10^{-11}. The inset shows the original curves while the main figure shows them rescaled by (7).

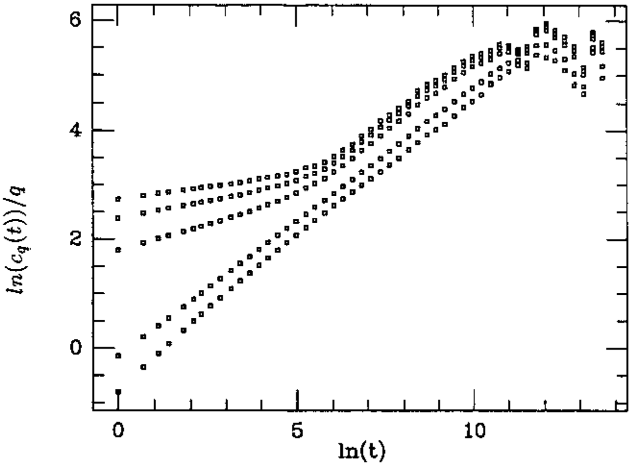

Figure 3. The qth-order correlation function for $\mu = 2$, $k = 10^{-10}$ and $q = 1, 2, 5, 7, 9$.

We studied the μ dependence of the correlation functions in this model. Figure 4 shows the scaling of $c_6(t)$ for different values of μ. The conclusion one can draw from this figure is the following: the length of the multi-affine scaling region seems to depend very much on the value of μ. In particular, for increasing μ the multi-affine region seems to disappear and the self-affine part tends to dominate.

This conclusion is in concordance with the fact that for large μ values the noise distribution approaches the bounded noise, so the multi-affine part should disappear. The mechanism of the transition from multi-affine to self-affine behaviour is very interesting here: the variations in μ affect the length of the scaling region and slightly after the value of the exponents (for large q values the variation of the H_q with μ is very small; this is why it cannot be observed on this figure). In figure 4 we have shifted vertically the curves for different μ values to see the difference in the scaling regions. In order to explain this phenomena, we have to study the effect of the rare events on the correlation functions.

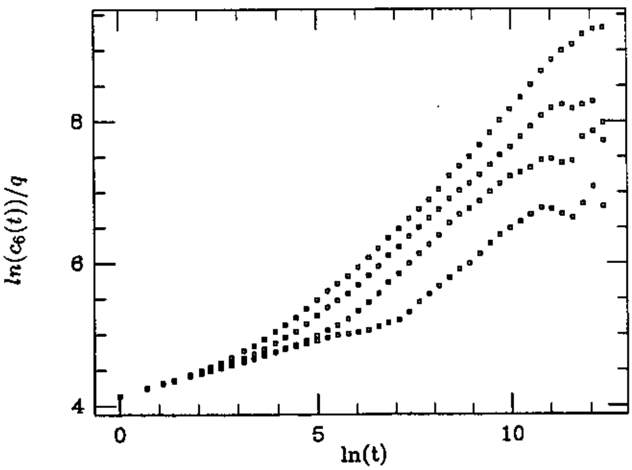

Figure 4. The sixth-order correlation function $c_6(t)$ for $\mu = 2, 3, 4, 5$ (from right to left).

Because the fluctuations take place in a very narrow region compared with $h^* \sim k^{-1/2}$, the response in $\Delta h_t = |h_{t+1} - h_t|$ is linear in the noise η. (This linearity has been verified numerically also for the rare events observed in the iteration. For rare events larger than k^{-1} a saturation phenomenon takes place, but the frequency of such a rare events is of order $k^{\mu+1}$ so for small k values it has no effect on the scaling.) If a large Δh_1 appears, the perturbation will decrease in time, converging linearly to the mean value of the fluctuations. Thus a perturbation Δh_1 has an effect of length τ in time, where $\tau \sim \Delta h_1 \sim \eta$. It can be easily shown that a perturbation with characteristic length τ affects the scaling of the correlation function $c_q(t)$ only if $\tau > t$. This assumption is valid only if the perturbation is alone, i.e. no more large η appear in the time interval τ. In the time interval t the expected value of the large fluctuations scales like [17, 18] $\eta_{\max} \sim t^{1/\mu}$, so it affects the scaling of the $c_q(t)$ only if $t^{1/\mu} > t$ which predicts a critical timescale $t_c \simeq (\text{constant})^{\mu/(\mu-1)}$. This relation predicts that for $t > t_c$ the rare events do not affect the scaling of the correlation function, so the cumulative affect of the small fluctuations will dominate. With increasing μ the multiaffine scaling region decrease, as has been observed on figure 4. We expect that the constant appearing in the relation for t_c has some k dependence, as has been observed in [12]. The arguments presented do not give the explicit form of this dependence; more detailed numerical studies are needed to investigate this problem.

Since the position of the transition point t_c cannot be extracted exactly from figure 4, we have only a qualitative agreement between the relation for t_c and the numerical results. Larger simulations are needed to verify numerically this relation.

Figure 5 shows the H_q spectrum measured for $\mu = 3$.

In [12] arguments showing a phase transition at $q = \mu$ were presented. The same arguments can be followed here also: the height differences are distributed according to the function [19] $P_n(\eta)$ so the qth moments diverge for $q > \mu$.

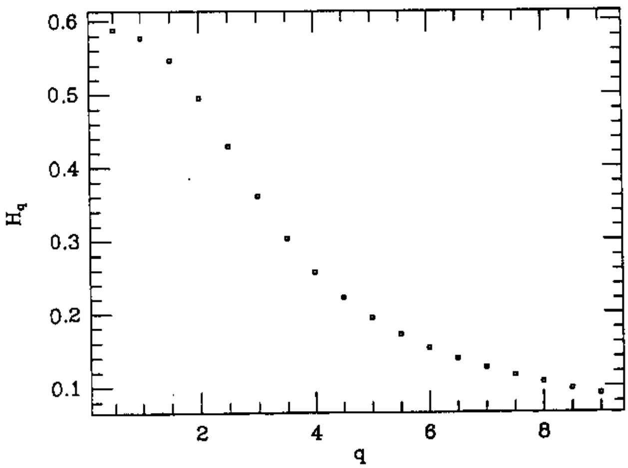

Figure 5. The H_q spectrum for $\mu = 3$.

At the end of this letter we should like to discuss the connections of the results obtained on this map with the real surface growth phenomena.

Although related to the KPZ equation, the approximations used in this paper may neglect some important phenomena affecting the surface growth. In particular we considered that the surface can be roughly approximated by a paraboloid of width L (which is taken to be constant) and height h_t and that the evolution in time affects only its height. Really an important role is played by the lateral growth of these paraboloid segments.

In the growth models for power law distributed noise a μ dependence of the scaling exponents α and β has been observed. On the other hand, a noise independence of the transient part expressed by (7) characterizes the map.

But with these differences we believe that the coupling observed between the noise and the deterministic smoothing dynamics is very interesting and might help us to understand the same type of phenomena occurring in surface growth. A non-trivial fixed point is introduced by the noise η. Starting form $h_0 = 0$ the height h_t approaches this fixed point following the scaling relation (7). After the saturation is achieved, in the steady-state state, h_t fluctuates, its mean value scaling as $k^{1/2}$. The fluctuations are the result of the convergence of the h_t to the fixed point, which changes its position randomly. The correlations in these fluctuations are very sensitive to the noise distribution (which governs the random motion of the fixed point): for power law distributed noise the time series shows multi-affine correlations, while for bounded noise they are simply self-affine. It is fascinating that qualitatively the same type of scaling has been observed in this model and in the spatial fluctuations of the rare-events-dominated roughening. In particular the same type of transition from multi-affine to self-affine scaling has been observed. We believe that these transitions have the same explanation: the insensitivity of the correlation functions $c_q(t)$ (or $c_q(x)$) to the fluctuations shorter in time (or space) than t (or x).

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