Physica A 178 (1991) 17–28
North-Holland

Multifractal spectra of multi-affine functions

Albert-László Barabási

Department of Atomic Physics, Eötvös University, P.O. Box 327, Budapest, 1445 Hungary

Péter Szépfalusy

Institute for Solid-State Physics, Eötvös University, P.O. Box 327, Budapest, 1445 Hungary and Central Research Institute for Physics, P.O. Box 49, Budapest, 1525 Hungary

Tamás Vicsek

Department of Atomic Physics, Eötvös University, P.O. Box 327, Budapest, 1445 Hungary and Institute for Technical Physics, P.O. Box 76, Budapest, 1325 Hungary

Received 15 June 1991

Self-affine functions $F(x)$ with multiscaling height correlations $c_q(x) \sim x^{qH_q}$ are described in terms of the standard multifractal formalism with a modified assumption for the partition. The corresponding quantities and expressions are shown to exhibit some characteristic differences from the standard ones. According to our calculations the $f(\alpha)$ type spectra are not uniquely determined by the H_q spectrum, but depend on the particular choice which is made for the dependence of N on x, where N is the number of points over which the average is taken. Our results are expected to be relevant in the analysis of signal type data obtained in experiments on systems with an underlying multiplicative process.

1. Introduction

Self-affine [1] fractals have been shown to be useful not only in describing the surfaces of clusters generated in various growth models like ballistic deposition or the Eden model [2], but they can be applied in the analysis of a number of fractal growth phenomena [3, 4] of practical importance, including thin film growth by vapor deposition, two phase viscous flow in porous media, formation of biological patterns and sedimentation of granular materials [5]. In addition, the time dependence of experimentally recorded quantities (signals) may exhibit self-affine scaling as well.

In nature many processes lead to fractal measures [6–8], which are characterized by an infinite hierarchy of fractal dimensions. In the multifractal formalism the fractal support and the measure defined on it is thought of as consisting of many interwoven fractal sets, all of them characterized by their own exponent α, and the fractal dimension of these sets being $f(\alpha)$.

Recently the idea of more than one characteristic exponent has been applied to self-affine functions, and it has been shown [9] that the qth order height–height correlation function may exhibit a non-trivial multiscaling behaviour characterized by a continuously changing exponent H_q. These functions will be called multi-affine fractals.

In this paper we discuss a modified way of extending the standard multifractal formalism to the description of multi-affine fractals in order to obtain a more complete characterization of their scaling properties. In section 2 we deal with continuous self-affine functions and calculate the $h(\gamma)$ spectrum, which is an analogue of the well known $f(\alpha)$ spectrum. We shall show that the relation between H_q and $f(\alpha)$ depends on the choice for the partition. At the end of the section we shall apply these results to a deterministic model which was found to exhibit multi-affine behaviour. The conclusions are given in section 3.

2. Continuous multi-affine functions

2.1. General formalism

According to their definition, single valued self-affine functions of a single variable satisfy the relation

$$F(x) \simeq \lambda^{-H} F(\lambda x) , \tag{1}$$

where λ is a parameter and H is the roughness or Hölder exponent [1, 10, 11]. For stochastic systems (1) holds only statistically. Alternatively, the height–height correlation function

$$c(x) = \langle |F(x') - F(x' + x)| \rangle_{x'}$$

of a self-affine function scales with x as $c(x) \sim x^H$ as $x \to 0$.

There are systems, however, whose height–height correlation function cannot be characterized by a single roughness exponent and for a more complete description one has to introduce an infinity hierarchy of characteristic exponents [7, 9]. This can be done by the introduction of the qth order height–height correlation function, defined as

$$c_q(x) = \frac{1}{N} \sum_{i=1}^{N} |F(x_i) - F(x_i + x)|^q , \tag{2}$$

where $N \gg 1$ is the number of points over which the average is taken. For a continuous signal $F(x)$, which without loss of generality can be considered as $x \in [0, 1]$, $c_q(x)$ can be computed as follows: Make a partition of the $[0, 1]$ interval in N equal parts and sum the qth powers of the height differences $\Delta F = |F(x_i) - F(x_i + x)|$. For this partition $x \sim 1/N$. However, a more general approach requires that the limits $x \to 0$ and $N \to \infty$ are taken independently. (Earlier studies [7] implicitly assumed the special case $x \sim 1/N$.) In the limit $x \to 0$ (which corresponds to $N \to \infty$) one expects that

$$c_q(x) \sim x^{qH_q}, \tag{3}$$

where H_q is a continuously changing exponent with q for multi-affine objects and it is constant for ordinary self-affine sets. Note that in our case qH_q is the natural choice for the exponent in the right-hand-side of eq. (3) (instead of $(q-1)H_q$ as for fractal measures).

As mentioned above we shall assume that, when evaluating (2), x and N may be related in a way different from $x \sim 1/N$. Let us consider the dependence $N \sim x^{-\phi}$. For the commonly used partition we have $\phi = 1$. For $\phi < 0$ in the $x \to 0$ limit $N \to 0$, which violates the condition that N has to be large. The case when the number of points N over which the average is taken is fixed, is equivalent to $\phi = 0$. The formalism for this special value of ϕ will be discussed in the appendix. Throughout this paper *we shall be concerned with the $\phi > 0$ case* when $N \to \infty$ for $x \to 0$. The choice of a particular partition has no effect on the H_q spectrum. However, as will be shown, ϕ enters the relations between the multifractal spectra.

In many cases nontrivial multiplicative processes can generate nonuniform surfaces having infinitely many singularities. These singularities appear in the scaling of ΔF in the vicinity of different points on the signal, and this scaling can be different from point to point [7]. This property can be described by an exponent γ_i through the following relation:

$$|F(x_i) - F(x_i + x)| \sim x^{\gamma_i} \tag{4}$$

in the limit $x \to 0$.

The noninteger exponent γ corresponds to the strength of the local singularity of the signal. Although γ depends on the actual position, there are many intervals of size x with the same index γ, and their number is expected to scale with x as

$$N_{\gamma_i}(x) \sim x^{-h(\gamma_i, \phi)}, \tag{5}$$

where $h(\gamma, \phi)$ for $\phi = 1$ is viewed as the fractal dimension of the subset of the points with the same exponent γ. For other ϕ values $h(\gamma, \phi)$ simply describes the scaling of the number of places with a given local singularity.

In analogy with the standard multifractal formalism [7, 8], in the limit $N \to \infty$ the correlation function (2) can be written as

$$c_q(x) \simeq \frac{1}{N} \int_{(\gamma')} x^{\gamma' q} x^{-h(\gamma', \phi)} \rho(\gamma') \, \mathrm{d}\gamma' .$$

For continuous systems with the above described partition for $x \to 0$, $N \sim x^{-\phi}$, so the above integral will be dominated by the value γ' which makes $\phi + \gamma' q - h(\gamma', \phi)$ smallest, provided that $\rho(\gamma')$ is nonzero. Thus, we replace γ by $\gamma(q)$, which is defined by the external conditions

$$\frac{\mathrm{d}}{\mathrm{d}\gamma'} [\phi + q\gamma' - h(\gamma', \phi)]|_{\gamma' = \gamma(q)} = 0$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}\gamma'^2} [\phi + q\gamma' - h(\gamma', \phi)]|_{\gamma' = \gamma(q)} > 0 .$$

Then it follows that

$$h'(\gamma) = q , \tag{6}$$

$$h''(\gamma) < 0 , \tag{7}$$

and from comparison with (3)

$$qH_q = \phi + q\gamma(q) - h(\gamma(q), \phi) \tag{8}$$

and

$$\gamma(q) = \frac{\mathrm{d}}{\mathrm{d}q} (qH_q) . \tag{9}$$

The relations (8), (9) and (6), (7) are the main results of this paper for the continuous multi-affine functions. The meaning of (8) is that the $h(q)$ spectrum takes its maximum value at $\gamma(q = 0)$, and in this point its value is equal to $h(\gamma(q = 0), \phi) = \phi$. As ϕ is varied from ∞ to 0 parts of the $h(\gamma, \phi)$ spectrum become negative until finally $h(\gamma, \phi = 0) \leq 0$. Before proceeding to examples it is important to point out the differences and similarities of these relations with those used in the standard multifractal formalism for fractal measures to link the generalized dimensions D_q and the $f(\alpha)$ spectrum.

First, a similarity between the correlation function $c_q(x)$ and the generating function $\chi(q) = \Sigma_i\, p_i^q$ for a measure p_i is obvious; however the scaling of the latter is described by an exponent $(q-1)D_q$, where $(q-1)$ appears due to the normalized condition of p_i, but in the former such a normalization does not exist, since the scaling exponent is qH_q.

Another difference is the term ϕ in relation (8), which appears as a result of the scaling of N with x in the limit $x \to 0$. However, the general technique is very similar: like in the case of the multifractal formalism, H_q is related to $h(\gamma, \phi)$ by a Legendre transformation.

2.2. Normalized height difference distribution, $f(\alpha, \phi)$ spectrum

An $f(\alpha, \phi)$ spectrum can be also associated to the hierarchy of $\Delta F(x)$ values, by introducing a normalization. Consider the $[0, 1]$ interval partitioned into N intervals and consider the following measure:

$$p_i(x) = \frac{|F(x_i) - F(x_i + x)|}{\Sigma_{i=1}^{N} |F(x_i) - F(x_i + x)|} \,. \tag{10}$$

This can be viewed as a probability measure, satisfying $\Sigma_{i=1}^{N}\, p_i(x) = 1$ and $p_i(x) \geq 0$, and one can construct the corresponding generating function

$$\chi_q(x) = \sum_{i=1}^{N} p_i^q(x)$$

and introduce the corresponding exponent $D_q(\phi)$ through the expression $\chi_q(x) \sim x^{(q-1)D_q(\phi)}$ in the limit $x \to 0$. Using (10) the scaling of $\chi_q(x)$ can be written as

$$\chi_q(x) \sim \frac{\Sigma_i\, |F(x_i) - F(x_i + x)|^q}{(\Sigma_i\, |F(x_i) - F(x_i + x)|)^q} \sim x^{q(H_q - H_1) + (q-1)\phi} \,,$$

which gives the relation between the generalized dimensions $D_q(\phi)$ of the normalized height differences and the multi-affine exponents H_q:

$$(q-1)D_q(\phi) = q(H_q - H_1) + (q-1)\phi \,. \tag{11}$$

A special case of this relation was obtained in ref. [12]. Using the well known Legendre transformations linking D_q and the corresponding $f(\alpha)$ function,

$$(q-1)D_q = q\alpha(q) - f(\alpha(q))\,, \qquad \alpha(q) = \frac{\mathrm{d}}{\mathrm{d}q}\,[(q-1)D_q]\,,$$

$$q = \frac{\mathrm{d}}{\mathrm{d}q}\,f(\alpha(q))\,,$$

one finds with (8) the relations between the corresponding quantities:

$$\alpha(q,\phi) = \gamma(q) - H_1 + \phi\,, \qquad f(\alpha(q,\phi),\phi) = h(\gamma(q),\phi)\,.$$

The above relations show that there is only a small difference between the $f(\alpha,\phi)$ and $h(\gamma,\phi)$ spectra, the former is shifted to the right by a constant value $(\phi - H_1)$, but the shape and the range of values of f and h are unchanged during the transformation.

For $c_q(x)$ the jumps are not normalized so the monotonicity conditions which are valid for $D_q(\phi)$ do not hold for H_q. On the other hand, using the relation between $D_q(\phi)$ and H_q and from the two monotonicity conditions existing for fractal measures [13]

$$D_q(\phi) \leqslant D_{q'}(\phi)$$

if $q > q'$, and

$$\frac{q-1}{q}\,D_q(\phi) \leqslant \frac{q'-1}{q'}\,D_{q'}(\phi)$$

if $q < q'$ and $qq' > 0$, one finds that the following two functions have to be decreasing:

$$g_1(q) = \frac{q}{q-1}\,(H_q - H_1)\,, \qquad g_2(q) = H_q + \frac{\phi}{q}\,,$$

in the sense that $g_1(q) \leqslant g_1(q')$ if $q \geqslant q'$, and $g_2(q) \leqslant g_2(q')$ if $q \geqslant q'$ and $qq' > 0$.

2.3. *A deterministic multi-affine model*

Let us examine the above derived relations on an exactly solvable deterministic model introduced ref. [9]. The iteration procedure, which is a generalization of the construction [10] proposed by Mandelbrot, is demonstrated in fig. 1. In each step of the recursion the intervals obtained in the previous step are replaced by the properly rescaled version of the generator, which has the form of an asymmetrical z made of four intervals. During this procedure every interval is regarded as a diagonal of a rectangle becoming

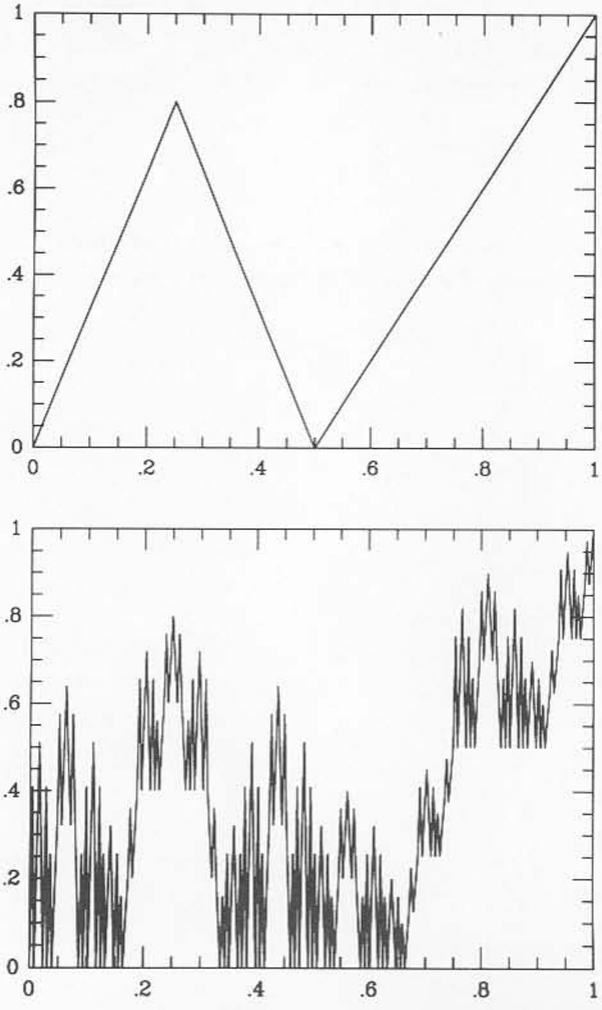

Fig. 1. Construction of a deterministic multi-affine function.

more and more elongated as the number of iterations k increases. The basis of the rectangle is divided into four parts and the generator replaces the intervals in such a way that its turnover are always at analogous positions (at the first generator and the middle of the basis). The function becomes multi-affine in the $k \to \infty$ limit. Depending on the parameter b_1 very different structures can be generated, the b_2 is fixed to be 0.5.

The H_q spectrum can be calculated for this construction exactly assuming that the scaling properties are entirely determined by the behaviour of the function over intervals of length 4^{-k}. Denoting with $N(\Delta h)$ the number of boxes in which $|h(x) - h(x + \Delta x)| = \Delta h$, we have $N(b_1^n b_2^{k-n}) = 2^k C_k^n$, where $n = 0, \ldots, k$. Thus $c_q(\Delta x) = \Sigma_{n=0}^k 2^{-k} C_k^n b_1^{nq} b_2^{(k-n)q}$ with $\Delta x = 4^{-k}$. Since this equation can be written as $c_q(\Delta x) = [\frac{1}{2}(b_1^q + b_2^q]^k$ we have

$$H_q = \frac{\ln[\frac{1}{2}(b_1^q + b_2^q)]}{q \ln(\frac{1}{4})} . \tag{12}$$

In the present approach the roughness exponent introduced earlier is $H = H_1$. In ref. [9] we discussed the aspects of numerical computation of this quantity.

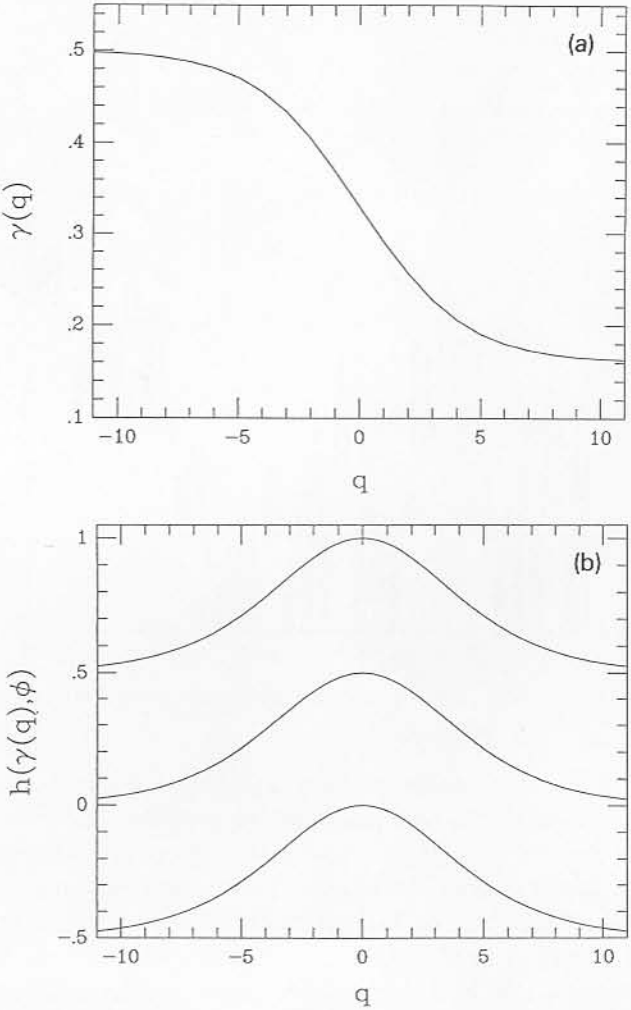

Fig. 2. Dependence of the exponents $\gamma(q)$ (a) and $h(q)$ (b) on q (in (b) $\phi = 1.0$, 0.5 and 0.0 from top to bottom).

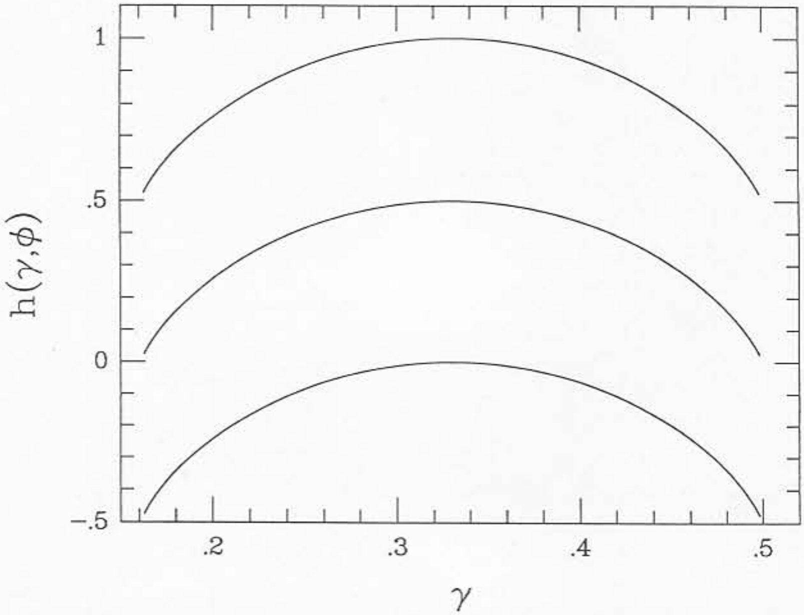

Fig. 3. The multifractal spectrum $h(\gamma, \phi)$ as a function of the local strength of the singularities γ for three selected values of $\phi = 1.0$, 0.5 and 0.0 (from top to bottom).

Relations (8) and (9) with (13) allow us to plot $\gamma(q)$ and $h(q)$ (fig. 2) as well as $h(\gamma, \phi)$ in fig. 3.

It is simple to make the normalization for the given model, providing the possibility of the computation of the corresponding $D_q(\phi)$ spectrum through relation (11). Using the usual $f(\alpha)$ formalism one can compute through Legendre transformations the corresponding $f(\alpha, \phi)$ spectrum (figs. 4a and b).

3. Conclusions

We have investigated the multifractal aspects of self-affine functions with a non-trivial spectrum of exponents characterizing the scaling behaviour of the moments of its height–height correlation function. With some modifications of the standard multifractal formalism the corresponding multifractal spectra and the expressions among them have been obtained. Self-affine functions and fractal measures exhibit several different characteristics, thus our expressions are not exactly the same as those of the standard multifractal formalism. On the other hand, the absolute value of the derivative of a multi-affine function is expected to behave the same way as a fractal measure; this is why the standard approach can be easily applied to the affine case.

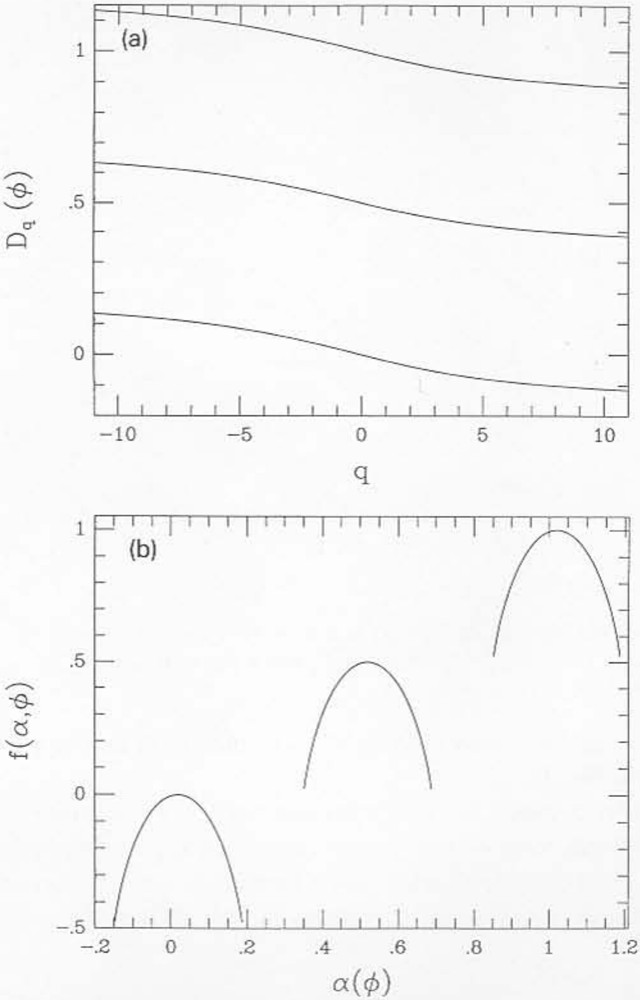

Fig. 4. If the distribution of height differences is normalized, the multifractal spectra $D_q(\phi)$ and $f(\alpha, \phi)$ obtained in our approach (shown in parts (a) and (b), respectively) are in complete analogy with the corresponding spectra of the standard multifractal formalism for $\phi = 1.0$. This figure demonstrates the differences for other values of ϕ (which are the same as in figs. 2 and 3).

We have shown that for multiaffine functions the $f(\alpha)$ type multifractal spectra are not uniquely defined; they depend on the partition which is used during the procedure of establishing relationships among the spectra. In particular, if we assume that $N \sim x^{\phi}$, the exponent ϕ is a simple additive term in the expressions.

Our results are expected to be relevant in the analysis of various kinds of data related to systems with an underlying multiplicative process. A typical application is the analysis of the results for the distribution of the passive scalar [14] or the velocity [15] in experiments on turbulent flows. In these experiments the spatial or temporal dependence of the actually measured quantities can be considered as a multi-affine function. This fact has been indicated by calculations of the moments of various quantities related to the coarse grained derivative of the above-mentioned distributions.

Acknowledgement

The present research was financially supported by the Hungarian Scientific Research Foundation Grant No. 693.

Appendix

For $\phi = 0$ the number of points N over which the average is taken in (2) is constant if our multiaffine function is considered only on the unit interval. On the other hand, for a complete multifractal analysis it is required that $N \to \infty$. This can be achieved by considering the function on the whole axis and taking the limit $x \to \infty$ in our expressions (instead of $x \to 0$). The reason for discussing this case is that in the experiments it is typical to record a continuously changing quantity at discrete time steps or at discrete positions in space.

Changing the limit $x \to 0$ to $x \to \infty$ does not result in any change in the formulae involving the multifractal spectra H_q and $h(\phi)$ only. In fact, in ref. [9], it was shown that for the deterministic construction of fig. 1 H_q is the same, either it is calculated for $\phi = 1$ exactly, or for $\phi = 0$ numerically. However, if we take the $x \to \infty$ limit, the expressions connecting the multifractal spectra D_q and H_q are changed. This is due to the fact that in this case the correct form of the expression $\chi_q(x) \sim x^{(q-1)D_q(\phi)}$ used in the limit $x \to 0$ is

$$\chi_q(x) \sim x^{-(q-1)D_q(\phi)} ,$$

where an extra minus sign had to be introduced. Thus, the relations presented in section 2.2 remain the same except that D_q has to be replaced by $-D_q$.

The case when both the function and the sampling points are discrete is qualitatively different. An example for such a signal is the distance of a randomly diffusing particle on a one-dimensional lattice as a function of time. Numerical simulation and scaling arguments show [9] that in this case the behaviour of H_q for $q < -1$ is entirely dominated by the existence of the smallest possible jump in the function. Obviously, our formalism, which is strictly for continuous signals with no cutoff for small differences, is not applicable to this situation.

References

[1] B.B. Mandelbrot, The Fractal Geometry of Nature (Freeman, San Francisco, 1982).
[2] P. Meakin, CRC Rev. Solid State Phys. 13 (1987) 143.
[3] H.E. Stanley and N. Ostrowsky, eds., Random Fluctuations and Pattern Growth (Kluwer, Dordrecht, 1988).
[4] T. Vicsek, Fractal Growth Phenomena (World Scientific, Singapore, 1989).
[5] F. Family and T. Vicsek, Dynamics of Fractal Surfaces (World Scientific, Singapore, 1991).
[6] B.B. Mandelbrot, J. Fluid Mech. 62 (1974) 331.
[7] U. Frisch and G. Parisi, in: Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics, M. Ghil, R. Benzi and G. Parisi, eds. (North-Holland, Amsterdam, 1985).
[8] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia and B. Shraiman, Phys. Rev. A 33 (1986) 1141.
[9] A.-L. Barabási and T. Vicsek, to be published.
[10] B.B. Mandelbrot, Phys. Scr. 32 (1985) 257.
[11] R.F. Voss, in: The Science of Fractal Images, H.O. Peitgen and D. Saupe, eds. (Springer, Berlin, 1988) ch. 1.
[12] C. Meneveau and K.R. Sreenivasan, Nucl. Phys. B (Proc. Suppl.) 2 (1987) 49.
[13] C. Beck, Physica D 41 (1990) 67.
[14] P.R. Prasad, C. Maneveau and K.R. Sreenivasan, Phys. Rev. Lett. 61 (1988) 74.
[15] M. Nelkin, J. Stat. Phys. 54 (1989) 1, and references therein.