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Multifractality of growing surfaces

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We have carried out large-scale computer simulations of experimentally motivated (1+1)-dimensional models of kinetic surface roughening with power-law-distributed amplitudes of uncorrelated noise. The appropriately normalized qth-order correlation function of the height differences $c_q(x)=\langle|h(x+x')-h(x')|^q\rangle$ shows strong multifractal scaling behavior up to a crossover length depending on the system size, i.e., $c_q(x)\sim x^{qH_q}$, where H_q is a continuously changing nontrivial function. Beyond the crossover length conventional scaling is found.

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The concept of multifractality has provided us with a deep insight into the complex nature of distributions and geometry associated with numerous important phenomena ranging from turbulence through diffusion-limited aggregation to crack formation. The infinite set of nontrivial exponents characteristic for a specific multifractal measure yields a much more appropriate description of fractals than the fractal dimension alone [1–3]. The discovery of multifractal properties has been helpful in the understanding of the structure and formation of various fractal objects; in particular, it played an essential role in the field of growth phenomena [4,5]. So far the attention has been focused in this respect mainly on self-similar objects although the multifractal formalism has been worked out for self-affine structures as well, leading to the concept of "multiaffinity" [6,7]. The purpose of the present paper is to demonstrate that experimentally motivated models of kinetic surface roughening obey multiaffine scaling.

Kinetic surface roughening is a phenomenon important for both science and technology [8]. Vapor deposition, growth of bacterial colonies, or fluid displacement can be mentioned as typical experimental realizations. The surface usually grows from a d-dimensional flat substrate of linear size L and, due to the presence of noisy excitations, gets rough during its evolution.

The development of the resulting surface can be well interpreted in terms of dynamic scaling [9] and self-affine fractal geometry [10]. In this approach the width w of the surface $h(x,t)$ is expressed as

$$w(t,L)\equiv[\langle h^2(x,t)\rangle-\langle h(x,t)\rangle^2]^{1/2}\sim t^\beta g(t/L^{\alpha/\beta}),$$

where g is a scaling function and the exponents α and β correspond to the algebraic behavior of the surface width as a function of space (for long times) and time (for short times), respectively. Similarly, for the height-height correlation function one has

$$c_2(x,t)=\langle[h(x',t')-h(x'+x,t'+t)]^2\rangle_{x',t'}$$
$$\sim t^{2\beta}f(x/t^{\beta/\alpha}).$$

A powerful theoretical approach to kinetic roughening is represented by the so-called Kardar-Parisi-Zhang (KPZ) equation [11] which describes the temporal development of the height variable $h(x,t)$,

$$\frac{\partial h}{\partial t}=\nu\nabla^2 h+\lambda(\nabla h)^2+\eta, \quad (1)$$

where ν is an effective surface tension and λ is the strength of lateral growth. The term η is usually assumed to be uncorrelated and bounded (e.g., Gaussian) noise. This equation seems to describe properly diverse computer models of kinetic roughening like ballistic deposition or Eden models; in particular, in 1+1 dimensions the dynamic renormalization-group solution turns out to be exact ($\alpha=\frac{1}{2}$ and $\beta=\frac{1}{3}$).

Recent experiments on quasi-two-dimensional two-phase viscous flows [12–14] and bacterial colony growth [15], however, have lead to exponents different from these: The results for α are in the range 0.75–0.85. As a possible resolution of this problem (the appearance of new universality classes) it has been suggested [16] that a power-law distribution of the noise amplitudes $P(\eta)\sim\eta^{-(1+\mu)}$ could lead to new exponents (roughening dominated by rare events). In fact, the μ dependence of the exponents α and β has been observed in numerical simulations both in 1+1 and 2+1 dimensions [16,17]. Moreover, the determination of the noise amplitudes from the reanalysis of experimental data seems to support this suggestion [14]. It is now crucial to understand the structure of the surface and the way it is built up in the presence of power-law-distributed noise.

In analogy with the case of self-similar fractals, the concept of multifractality can be useful for rough surfaces as well. In this context, it has been suggested recently [6] that for a class of surfaces the qth-order height-height correlation functions should be studied which, for fixed time t in 1+1 dimensions, are expected to exhibit the following scaling:

$$c_q(x,t)=\frac{1}{L}\sum_{i=1}^{L}|h(x_i,t)-h(x_i+x,t)|^q\sim x^{qH_q}, \quad (2)$$

where H_q is an exponent continuously changing with q. Surfaces with height correlations satisfying (2) are called "multiaffine" and using the multifractal formalism relating the local singularities of the surface to the H_q spectrum, they can be described in terms of multifractality [7]. Expression (2) is in analogy with the qth-order velocity structure functions of fully developed turbulence which has been found to exhibit a nontrivial q-dependent scaling both experimentally [18] and in model calculations [19].

In this paper we present numerical evidence of multiaffine scaling in the model [16] of rare-events-dominated roughening. The scaling behavior (2) demonstrated here for a kinetic growth phenomenon implies that the distribution of the height differences can be described in terms of multifractal spectra. This multifractality is specific to rare-events-dominated roughening; growth models obeying the KPZ equation with bounded (e.g., Gaussian) noise results in a constant H_q.

We have simulated in 1+1 dimensions the evolution of the continuous height variable $h(x,t)$ subject to some local uncorrelated noise $\eta(x,t)$ using a discrete model in both space and time. Starting from a flat interface $h=0$ at $t=0$, the system evolves up to time t, performing t times the following rules: (i) The noise $\eta(x,t)$ is added to every site; (ii) each site takes a new value $h(x,t+1)$ equal to the maximum of itself and its two neighbors.

We used periodic boundary conditions for the space dimension. This model is fully parallel in nature and can be implemented efficiently on the Connection Machine (CM-2) and it has been used successfully to investigate the effect of rare events on the exponents α and β. The noise η was taken from a power-law distribution of the form

$$P(\eta)\sim\frac{1}{\eta^{1+\mu}} \text{ for } \eta>1;\; P(\eta)=0 \text{ otherwise}. \quad (3)$$

We have chosen $\mu=3$ for which the change in the roughening exponent is known to be significant (α is in the range 0.7–0.8 instead of $\frac{1}{2}$). The computations were performed in single precision (32 bits) floating-point arithmetic except for the critical part of the generation of the power-law-distributed noise. The program performed 8×10^6 updates per sec on a 16-kbyte processor CM-2 without double-precision hardware.

First we demonstrate the multifractal analysis of a single run for a system of size $L=131\,072$ (2^{17}). Figure 1 presents the qth root of the qth-order correlation functions ($q>0$) on log-log plots for two different stages of the growth: $t=10$ and 1000. In the absence of multifractal scaling one would expect two regimes for a given time in such a plot. In the first one—up to $x\sim t^{\beta/\alpha}$—the correlations have already been developed, resulting in parallel lines with the slope α. Beyond $x\sim t^{\beta/\alpha}$ no correlations are present and, correspondingly, the graphs cross over into horizontal lines. The fact that in the rare-events-dominated model regions with q-dependent slopes are

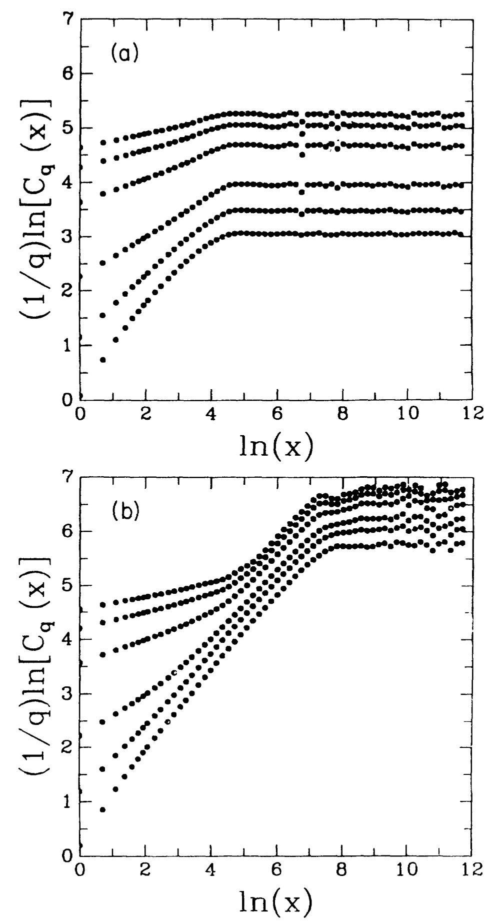

FIG. 1. The qth-order correlation functions at different stages of the growth: (a) after 10 sweeps and (b) after 1000 sweeps. On all figures $q=1$, 2, 3, 5, 7, and 9, increasing from the bottom to the top.

present shows that multifractal scaling characterizes the behavior. In the early stage of growth [$t=10$, Fig. 1(a)] typical multifractal scaling can be observed: The qth-order correlation functions are described by q-dependent exponents H_q. As the growth proceeds further an interesting crossover occurs and the lines due to the different order correlation functions become essentially parallel [Fig. 1(b)] with about the same nontrivial slope corresponding to the roughening exponent $(\alpha(\mu))$.

Figure 2 presents data for $L=65\,536$ at time $t=602\,890$, i.e., deep in the saturation regime; an average over 5 runs was taken. This is the most relevant set of our results. The important conclusions one can draw from this figure are the following: (i) The initial part of the data sets for each q exhibits scaling behavior with a unique slope depending on q, i.e., multifractal scaling is present; (ii) this kind of scaling crosses over into the uniform scaling behavior for x exceeding some characteristic crossover length $x_\times$.

In Fig. 3 the function qH_q is presented as measured in the multifractal scaling region. The deviations from the simple scaling behavior are clear; changes in qH_q as a function of q are rather dramatic. In fact, our scaling considerations to be discussed below indicate that there should be a phase-transition-like feature in the H_q spectrum at a μ-dependent q value.

Feature (ii) means that in addition to the characteristic length $\sim t^{\beta/\alpha}$ always present in kinetic roughening until the system size L is reached at saturation, a new characteristic length $x_\times$ occurs in surface growth dominated by rare events. For $x<x_\times$ the qth-order correlations show multifractal scaling behavior. For $x_\times < x < t^{\beta/\alpha}$ conventional scaling sets in while no correlations are present for $x > t^{\beta/\alpha}$.

The situation with the new crossover length is somewhat similar to the case where the intrinsic surface width influences the behavior [20] The intrinsic width consists usually of short-wavelength fluctuations and its development precedes in many kinetic roughening processes the build up of the long-wavelength scaling fluctuations.

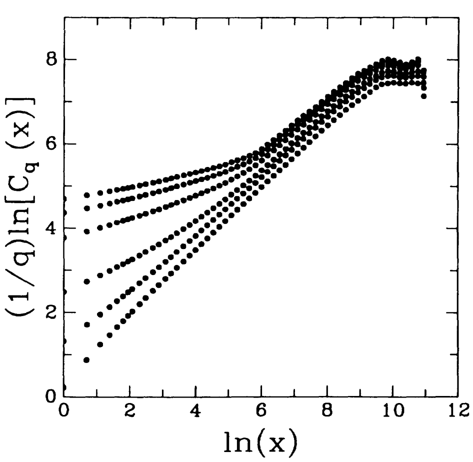

FIG. 2. The qth-order correlation functions after the surface width has saturated ($L=2^{16}$ and at $t=602\,890$ sweeps) for the same values of q as in Fig. 1. An average over 7 runs was taken.

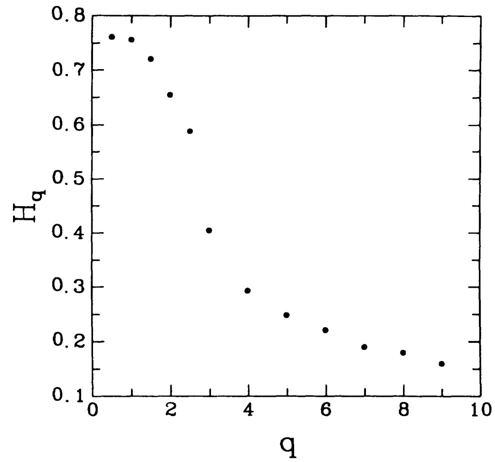

FIG. 3. The exponent H_q vs q as taken from runs described in Fig. 2. The sharp change at $q\sim 3(=\mu)$ can be an indication of a "phase transition."

However, there are important differences between the situation with intrinsic width and the crossover we observed for rare-events-dominated kinetic roughening. The usual intrinsic width does not obey scaling [21] and the crossover length below which it determines the behavior is independent of the system size L. Here we see that for $x<x_\times$ multifractal scaling is valid. Moreover, the crossover length $x_\times$ depends on L, as can be seen in Fig. 1.

The new characteristic length $x_\times$ is an immediate consequence of the rare events. A perturbation due to a large jump in the surface propagates in the lateral direction linearly until the rest of the surface catches up (see Fig. 1 in Buldyrev *et al.* [17]). In the mean-field approach [22] the surface width at saturation is identified with the characteristic large jumps, therefore the size of such jumps scales with the system size as $\sim L^{\alpha}$. Due to the linear propagation of the perturbation a characteristic length in the substrate direction occurs which is of the same order as the width. For distances smaller than this length the correlations are dominated by the large jumps [23], while on larger lengths the self-affine structure of the whole surface becomes apparent. This consideration implies [24] $x_\times \sim L^{\alpha}$ and this means that the relative size of the region over which the multifractal scaling behavior can be observed on a larger logarithmic scale becomes dominant in the large L limit. We have estimated the size dependence of the crossover length as $x_\times \approx 18$, 37, 115, and 245 for $L=2^{12}$, 2^{14}, 2^{15}, and 2^{16}, respectively. Although $x_\times$ is quite small, these estimates are consistent with the above scaling consideration.

An important feature of the considered qth-order correlation functions has to be pointed out. For small x [$x=O(1)$], the height differences are essentially distributed according to the function $P(\eta)$—at least as far as the large differences are concerned [25]. Since $P(\eta)$ is a power-law distribution, its qth moments diverge as $\sim N^{(q-\mu)/\mu}$ for $q>\mu$, where N is L times the number of samples. Therefore, these correlation functions have to be properly normalized by this factor. For finite systems and a finite number of samples one can always define them according to (2). However, the normalized correlation functions $c_q/N^{(q-\mu)/\mu}$ do exist in the thermodynamic limit.

It is natural to assume that the behavior for the diverging and nondiverging moments is qualitatively different; this is expected to lead to qualitative change in the H_q spectrum often described as a "phase transition" [26]. Figure 3 shows an indication of such a phase transition at $q \sim \mu = 3$.

We have demonstrated that the concept of multifractal scaling is very useful in describing kinetic roughening dominated by rare events. We could identify a crossover length below which the qth-order correlation functions showed multifractal scaling behavior. Beyond this length which increases with the system size, conventional scaling sets in. Furthermore, our preliminary results indicate that the exponents H_q do not depend on the system size or the time in the region we considered. Finally, we expect that the relatively sharp turn in the H_q spectrum corresponds to a phase transition at $q = \mu$.

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