ON CRISES AND SUPERTRACKS: AN ATTEMPT OF A UNIFIED THEORY *

A. -L. BARABASI,** L. NITSCH** I. A. DOROBANŢU***

** Faculty of Physics, University of Bucharest, Romania
*** Central Institute of Physics, Bucharest, POBox, MG-6, Romania

An attempt is presented to study from a unified point of view crises and supertracks. The concept of n-th order crisis is introduced and used to establish a general frame for describing the crises of one-dimensional maps.

The concepts of supertracks [1] and crises [2 –3] are probably the most interesting in the realm of the chaotic world. The former refers to orbits starting from the maximum point of a one-dimensional (ld) map; the latter considers situations arising during the collision of an unstable fixed point or a periodic orbit with an attractor, amounting to a sudden change of the respective attractor (which in fact explains the name of the phenomenon).

The usefulness of the supertracks (ST) and of the associated supertrack functions (STF) is related both to the possibility to extend their definition for a large class of one-dimensional maps [4] and to use them for explaining in a unified way interior as well as boundary crises [2, 3].

Our aim in this paper is to show that a general formulation in this

crises.

Similar to the use we made of STFs to calculate the position of periodic windows of the type RL^{n-2} and $RL^{n-3}R$ by means of scaling relations resulting from a renormalization scheme, we shall consider in the following maps of the interval of the same kind as those in Metropolis, Stein and Stein [5], Feigenbaum [6], i.e. with :

$$\text{a)} \quad x_{n+1} = F(\lambda, x_n) = \lambda f(x_n) \tag{1}$$

$f(x)$ single-valued, piecewise C^1 on [0,1], strictly positive on (0, 1), with $f(0) = f(1) = 1$; b) $f'(x^*)$ existing and being zero whenever $f(x^*) = f_{max}$; c) a λ_0 existing such that for $\lambda_0 < \lambda < \lambda_{max} = 1/f_{max}$, $F(\lambda, x)$ has only two fixed points, both repellent, of which one is the origin.

The n-th order STF, $S_n(\lambda)$ is then defined as

$$S_n(\lambda) = F^n(\lambda, x^*) \tag{2}$$

a definition which for quadratic maps turns into the Oblow expres on,

$$S_{n+1}(\lambda) = \lambda S_n(x)(1 - S_n(x)) \tag{3}$$

* Received March 11, 1988

In general,

$$S_0(\lambda) = x^*$$
$$S_1(\lambda) = \lambda f(x^*) = \lambda/\lambda_{max} \quad (4)$$
$$S_2(\lambda) = \lambda f(\lambda/\lambda_{max})$$

One finds out that:

1. The intersection of $S_n(\lambda)$ and $S_0(\lambda)$ gives the position (λ_n) of the n-th order supercycle:

$$S_n(\lambda_n) = S_0(\lambda_n) \quad (5)$$

2. $$S_{n+1}(\lambda) = F(\lambda, S_n(\lambda)) \text{ (cf.(2))} \quad (6)$$

3. For any $n \geqslant 2$, $S_n(\lambda_{max}) = 0$ (7)

A relation of the type (5) is important for, when it takes place, the n stable points of the $2n$ fixed points created by a tangent bifurcation, which generates an n-order periodic window, become superstable. Increasing further the system parameter, the stable orbit loses its stability and, after a cascade of period doublings, becomes chaotic. At this point, we observe as a typical feature the presence of n chaotic bands, evident on the computer plot of the bifurcation diagram and which are successively visited by the system. This stripped region is bounded by $S_{n-k}(\lambda)$ and $S_{2n-k}(\lambda)$, $k = 0, 1, \ldots, n-1$. A blow-up of any of these bands shows a bifurcation diagram similar to the original one. The stripped domain terminates when it is no longer bounded by the STFs S_{n-k} and S_{2n-k}. This takes place when

$$F^{2n}(\lambda_c, x^*) = x^+ \quad (8)$$

where x^+ is the unstable fixed point of $F^n(\lambda, x)$, generated by the tangent bifurcation, so that

$$F^n(\lambda, x^+) = x^+ \quad (9)$$

Equation (9) gives in principle the position of x^+ (the nearest unstable root of (9) to x^*). From what has been said right above, we can see that we do in fact describe a crisis situation, and the corresponding value λ_c is given by eqn. (8).

Now, equation (6) leads immediately to

$$S^{kn}(\lambda_c) = x^+, k \geqslant 2 \quad (10)$$

Therefore, a crisis point is found at the intersection of (kn)-th order STF$(k \geqslant 2)$ at λ_c. We also notice that for $k \geqslant 2$

$$\frac{dS_{2n}}{d\lambda} \cdot \frac{dS_{3n}}{d\lambda} > 0 \quad (11)$$

so that the slopes of the (kn)-th order STF have the same sign. Otherwise, the intersection of STFs defines a star point, an unstable singularity in the chaotic domain [1]. More precisely, a star is formed for $\lambda = \lambda_s$, where

$$S_p(\lambda_s) = S_{p+n}(\lambda_s) \quad (12a)$$

$$\frac{dS_p(\lambda_s)}{d\lambda} \cdot \frac{dS_{p+n}(\lambda_s)}{d\lambda} < 0 \quad (12b)$$

We shall say that at the point where a supertrack of order $2n$ crosses a supertrack of order $3n$, and no other crossing with a supertrack of order less than $3n$ exists at the same point, and $s^{2n}(\lambda) \neq x^*$ then we have:

— an n-th order crisis when the slopes have the same sign;

— an n-th order star, otherwise.

Star points are rather frequent due to complicated shapes of STFs. In this same region, particularly sensitive to the changes in the value of the parameter are the unstable n-cycles, which do not show bands of stability.

Returning now to crisis conditions (10) and (11), it is to be noted that in the case of 1d unimodal maps for $n = 1$ and $n \geq 2$ we have either a boundary or an interior crisis (the collision with the attractor takes place at the boundary or inside the basin of attraction, respectively). We see thus that STFs offer a general frame to describe both types of crisis occurring in 1d, unimodal maps.

In the remaining of the paper, we shall use STFs for 1d maps with two extrema (Figure 1), in particular for

$$F(\lambda, x) = -x^3 + (\lambda + 1)x \tag{13}$$

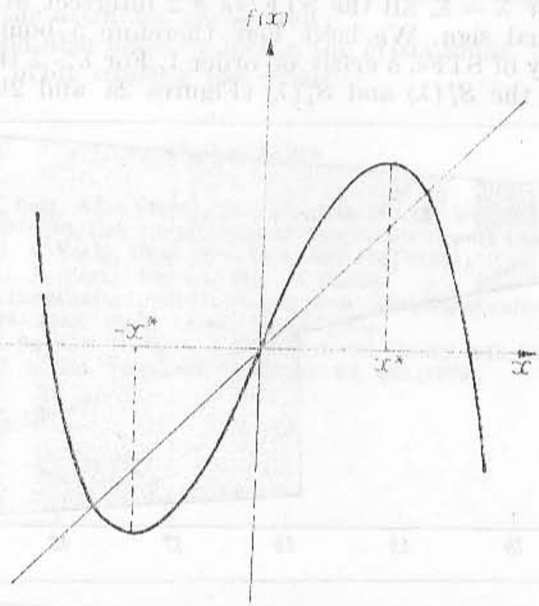

Fig. 1. — The map $F(\lambda, x)$ of eqn. (13).

The two extrema are here

$$\pm x^* = \sqrt{(\lambda + 1)/3} \tag{14}$$

Starting from an $x_0 > 0$ we easily find that the bifurcation diagram looks very strange (Figure 2a), showing a crisis at $\lambda = \bar{\lambda} = 3\sqrt{3/2} = 1.5980 \ldots$, where $\bar{\lambda}$ is the solution of

$$(\bar{\lambda} + 1)^{1/2} = f(x^*) \tag{15}$$

The crisis originates in the collision of the $x = 0$ unstable fixed points with the basin of attraction — it would thus seem to be a boundary crisis; it does however *not* destroy the attractor, a fact which as we shall shortly see is confirmed by the STF analysis. On the other hand, a new crisis, destroying now the attractor, is exhibited for $\lambda = 2$, but in this case it is very hard to locate the unstable orbit colliding with the attractor. Destruction of the attractor makes us suspect that in this case we have also to do with a boundary crisis.

For two extrema maps, two families of STFs can be defined starting from each of the extrema:

$$S_n^+(\lambda) = F^n(\lambda, x^*) \tag{16a}$$

$$S_n^-(\lambda) = F^n(\lambda, -x^*) \tag{16b}$$

Relation (6) holds in this case too, but other results are to be reformulated. We shall only discuss $S_n^+(\lambda)$, the case of $S_n^-(\lambda)$ being completely symmetric (with respect to the axis $x = 0$).

We thus notice that for $\lambda < \bar{\lambda}$ the chaotic domain is bounded by S_1^+ and S_2^+. For $\lambda = \bar{\lambda}$, all the STF$^+sn \geqslant 2$ intersect at $x = 0$, their slopes having equal sign. We have here therefore a boundary crisis or, in the terminology of STFs, a crisis of order 1. For $\lambda > \bar{\lambda}$ the chaotic domain is bounded by the $S_1^+(\lambda)$ and $S_1^-(\lambda)$ (Figures 2a and 2b). At $\lambda = 2$, the

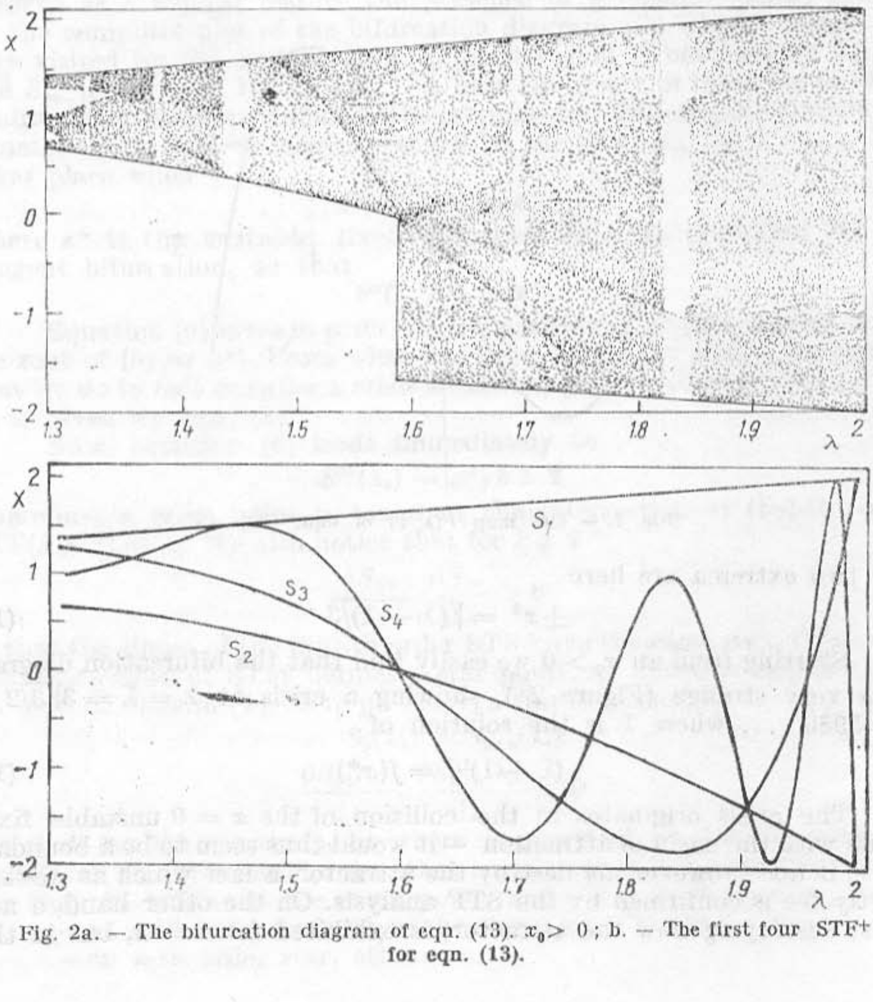

Fig. 2a. — The bifurcation diagram of eqn. (13). $x_0 > 0$; b. — The first four STF$^+$ for eqn. (13).

attractor is suddenly destroyed. At this point, all STFs of even/odd order cross at $S_n^+(2) = -2$ and 2, respectively. Condition (11) is satisfied for $n = 2$, so that we have a 2nd order crisis (a boundary one, in the terminology of Grebogi-Ott-Yorke), caused by the collision with an unstable 2 cycle. This indeed exists as an oscillation between positive and negative semiaxes of the attractor, i.e.

$$F(2, 2) = -2 \text{ and } F(2, -2) = 2 \tag{17}$$

In conclusion :

— we found that STFs allow a useful and more natural definition of crisis phenomena in 1d maps ;

— this new frame makes possible to explain a larger family of sudden changes in maps, in particular crises which look hard to identify in a chain of collisions with the attractor considered ;

— this method also helps to locate the crisis point, the period and the origin of the orbit causing the crisis.

REFERENCES

1. E. M. Oblow, Phys. Lett. A128 (1988) 406 ; Supertracks : the building blocks of chaos, ORNL/TM-10550, Oak Ridge National Laboratory Report (August, 1987).
2. C. Grebogi, E. Ott, J. A. Yorke, Phys. Rev. Lett., **48**, 1507 (1982).
3. C. Grebogi, E. Ott, J. A. Yorke, Physica, **7D**, 181 (1983).
4. A. -L. Barabasi, I. A. Dorobanţu, L. Nitsch, *Visiting chaos : a display of universalities*, Preprint, CIP-FT-349—1989, Phys. Letters A, in press.
5. N. Metropolis, M. L. Stein, P. R. Stein, J. Combinatorial Theory, **A15**, 25 (1973).
6. M. J. Feigenbaum, J. S. Tat. Phys., **19**, 25 (1978) ; **21**, 669 (1979).