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ANOMALOUS INTERFACE ROUGHENING: THE ROLE OF A GRADIENT IN THE DENSITY OF PINNING SITES

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Abstract

We study the effect on interface roughening of a gradient ∇p in the density of pinning sites p. We identify a new correlation length, ξ, which is a function of ∇p : $\xi \sim (\nabla p)^{-\gamma/\alpha}$, where $\alpha = \nu_\perp/\nu_\parallel$ is the roughness exponent, and $\gamma = \nu_\perp/(1+\nu_\perp)$. The exponents $\nu_\perp$ and $\nu_\parallel$ characterize the transverse and longitudinal correlation lengths. To investigate the effect of ∇p on the scaling properties of the interface in $(1+1)$ and $(2+1)$ dimensions, we calculate the critical concentration, p_c, and the exponents γ and α from which $\nu_\perp$ and $\nu_\parallel$ can be determined. Our results are in qualitative agreement with some of the features of imbibition experiments.

1. INTRODUCTION

Recently the growth of rough interfaces has attracted great interest, partly fueled by the broad interdisciplinary aspects of the subject.[1–5] Early studies[1,2,5] showed that the *rms* width of an interface, w, scales with time, t, and system size, L, as:

$$w(L, t) \equiv \langle [h(x, t) - \bar{h}(t)]^2 \rangle^{1/2} \sim L^\alpha f\left(\frac{t}{L^z}\right), \tag{1}$$

where $h(x, t)$ is the height of the interface and the function $f(x)$ satisfies $f(x \ll 1) \sim x^\beta$ and $f(x \gg 1) \sim const.$ The exponents α and $z \equiv \alpha/\beta$ are called the roughness and dynamical exponents, respectively.

Several authors[6–12] have recently suggested that for the case of an interface growing in a disordered medium, the roughening is caused by a quenched noise which is a function of the height of the interface but not of time. This proposed quenched noise not only enabled researchers to explain the results of some imbibition experiments in $(1+1)$ and $(2+1)$ dimensions,[7,8] but also resulted in the mapping of the growth of the interface to directed percolation and directed surfaces.[6–9]

In Ref. 7 (and similarly in Ref. 6) the interface stops growing, i.e., becomes pinned, when it encounters a directed percolating cluster of *pinning sites.* This fact allows the determination of the roughness exponent in $(1+1)$ dimensions. In directed percolation, the characteristic lengths in the directions normal and parallel to the interface scale as[13]:

$$\xi_\perp \sim |p_c - p|^{-\nu_\perp}\,, \quad \xi_{||} \sim |p_c - p|^{-\nu_{||}}\,. \tag{2}$$

The width of the pinned interface is proportional to $\xi_\perp$. When a directed path percolates the system, for $L \ll \xi_{||}$, we expect[6,7]:

$$w \sim \xi_\perp \sim \xi_{||}^{\nu_\perp/\nu_{||}} \sim L^{\nu_\perp/\nu_{||}}\,. \tag{3}$$

Thus,

$$\alpha = \nu_\perp/\nu_{||}\,. \tag{4}$$

From the calculated values of $\nu_\perp = 1.097 \pm 0.001$ and $\nu_{||} = 1.733 \pm 0.001$,[14,15] Eq. (4) predicts for the roughness exponent $\alpha = 0.633 \pm 0.001$.

In Ref. 7, a simple set of imbibition experiments was presented. Paper towels were dipped into a reservoir filled with various colored liquids (e.g. coffee, ink) and the propagating wetting front observed. The wetting front reaches a certain height above the level of the liquid and stops propagating when the evaporation of the liquid induces the *pinning* of the interface by the inhomogeneities of the paper. The rough boundary between colored and uncolored areas is digitized and a roughness exponent $\alpha \cong 0.63$ was found,[7] in agreement with the predictions of directed percolation.

However, three experimental features remain unexplained.

- (i) The scaling behavior was observed in Ref. 7 for length scales inferior to 1 cm only, much smaller than the system size;
- (ii) Although the model of Ref. 7 correctly reproduces the anomalously large value of α observed in some experiments,[7] it does not make clear which mechanism drives the experimental system towards the critical probability;
- (iii) The experiments show that the same fluid in the same paper reaches a higher level if the evaporation from the surface of the paper is reduced by enclosing the whole apparatus in a box or when the viscosity of the fluid is reduced by decreasing the concentration of coffee or ink (see Fig. 1).

To better understand the above experimental features, we offer the following considerations. It is reasonable to assume that the amount of liquid per unit area, $\rho(h)$, in the paper decreases with height, h, by the amount of evaporated liquid:

$$\frac{\partial \rho(h)}{\partial h} = -\frac{a}{v}\,, \tag{5}$$

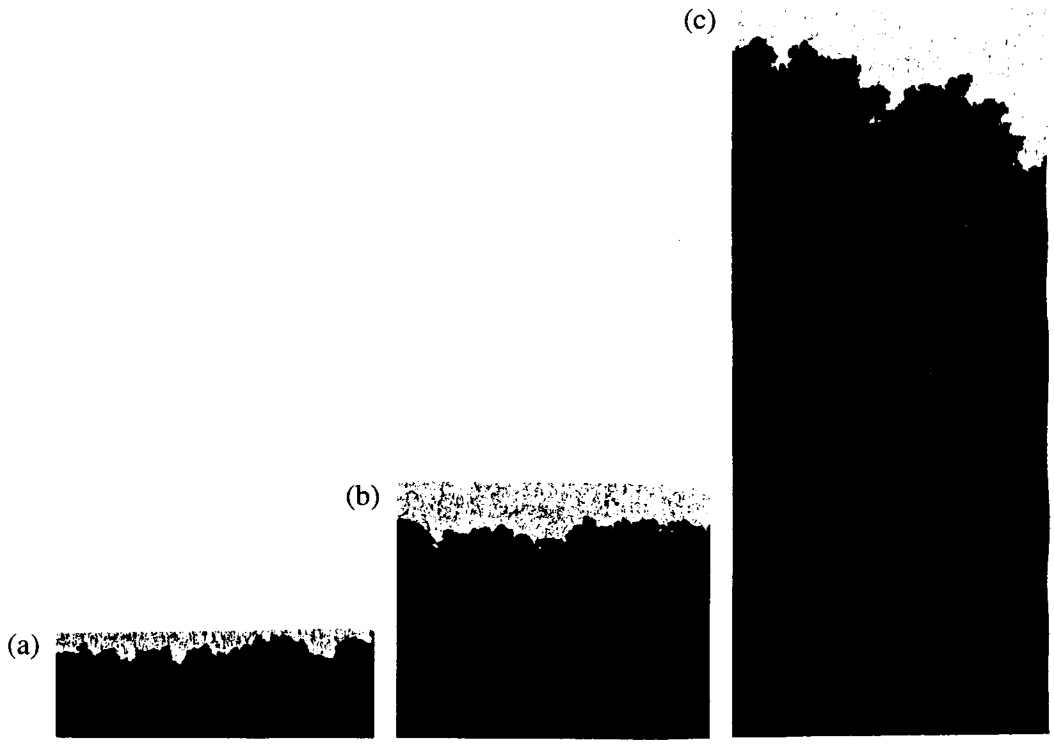

Fig. 1 The completely stopped interfaces in imbibition experiments with coffee and paper towels: (a) high evaporation rate, (b) medium evaporation rate, and (c) low evaporation rate.

where a is the evaporation rate (the amount of fluid that evaporates from unit area in unit time) and v is the velocity of the fluid particles in the paper. Assuming that a and v are *constants*, we can conclude that ρ is a linear function of the height:

$$\rho(h) = \rho(0) - \frac{a}{v}h\,. \tag{6}$$

The fluid in the paper propagates through fibers randomly distributed and connected. We assume that the fluid will penetrate from one fiber to the next only if there is a certain amount of accumulated liquid in the wet fiber. Since the amount of this accumulation depends on the properties of the connections between the fibers — which are very heterogeneous — the concentration of the *pinning connections* between fibers increases with the decrease of the amount of water in the paper, $\rho(h)$. The critical threshold p_c corresponds to the average height reached by the liquid in the paper during the final stage of the experiment when the interface ceases to propagate.

Since in the model the concentration of *pinning connections* between fibers corresponds to the density of pinning sites, the probability of a site being blocked should increase with height. If we assume this increase to happen with a constant gradient, $\nabla p > 0$, then in the experiment ∇p should be inversely proportional to the height of the completely stopped interface. Due to the gradient, a new correlation length ξ is found, in analogy with the theory of Sapoval, Rosso and Gouyet on gradient percolation.[16–18]. This correlation length can be related to the upper cutoff of scaling behavior in the experiments of Ref. 7.

2. THE MODEL

Consider a square lattice and distribute randomly interface pinning forces in the sites. For each site we compare the pinning force with the interface driving force. If the former is larger, the site is considered blocked (pinning site), otherwise it is empty, in which case the site can be invaded during the growth. By varying the driving force with the height we can produce a gradient in the density of pinning sites.

We start from a flat interface on the bottom edge of a lattice (with periodic boundary conditions), and allow the interface to grow by invasion of empty sites. In the growth process we move along all columns, from left to right, one after the other (see Fig. 3). For each column we first try to grow it, from its top position, one site up. If this is not possible (because that site is blocked), we try to grow it to the left and/or to the right *if and only*

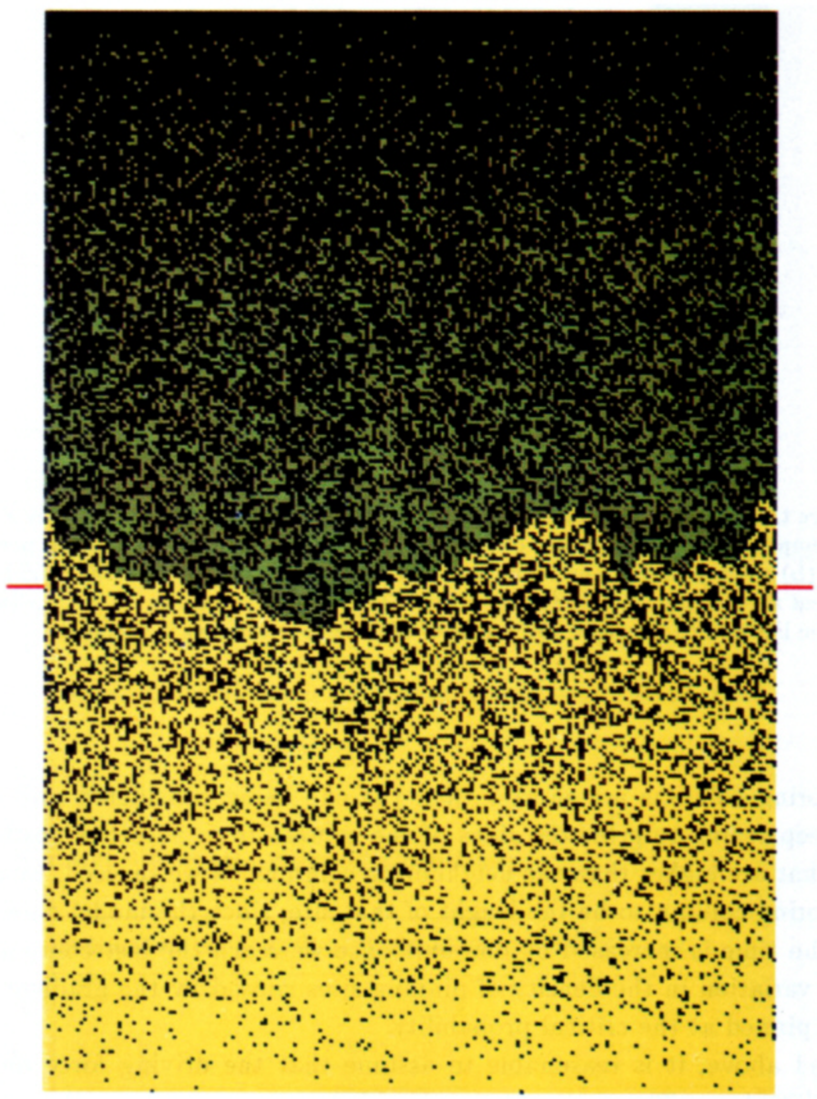

Fig. 2 We show here the pinned interface for a system with a constant gradient. Black dots represent pinning sites, green dots empty sites and yellow dots invaded sites. The red line shows the position of the critical probability. We can see that the interface stops when it reaches a percolating cluster of pinning sites, what occurs for $p = p_c$.

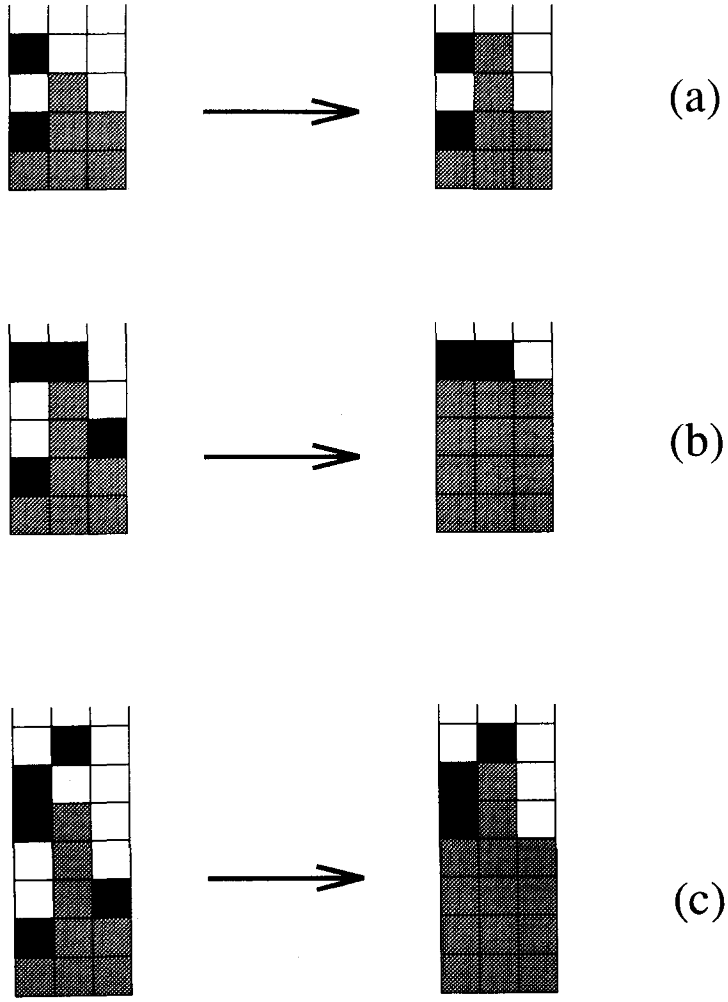

Fig. 3 Show are three cases of growth for the middle column with the conventions: grey sites are invaded, white sites are empty, and black sites are blocked (pinning) sites. In case **(a)** the top position grows one site up. In case **(b)** the top position grows both to the right and to the left, and the height of the neighbor column is updated accordingly. In case **(c)** the top position grows up and the second position from the top grows both to the left and to the right; the height of the neighbor columns is updated.

if the neighboring columns are shorter than the one being considered *and if* that site is empty. We keep trying to grow the sides of the column from all other positions, beneath the top one, that are higher than the neighboring columns. If we succeed, we set the height of the neighboring column to be the height of that site, since the model does not allow for overhangs. The time is increased by one unit once we reach the last column. We can see in Fig. 2, the variation in the density of pinning sites caused by the gradient and that the interface gets pinned at the critical probability.

As discussed above, it is reasonable to assume that the driving force depends on the height of the interface. The driving force should decrease monotonically with the height. This means that the density of pinning sites will increase monotonically. For example, in imbibition experiments the density of fluid, $\rho(h)$, decreases because of evaporation, and thus the driving force should also decrease.

The simplest choice [see Eqs. (5) and (6)] for the functional form of the driving force $f(h)$ is a linear function,

$$f(h) = f_0 - (\nabla p)h\,. \tag{7}$$

With this form of $f(h)$, we get the following density of pinning sites, $p(h)$:

$$p(h) = p_0 + (\nabla p)h\,. \tag{8}$$

Finding a suitable physical justification for this particular choice is difficult since the microscopic processes responsible for the interface growth and roughening are extremely complex. However, we shall find that the value of $p_0(< p_c)$ does not affect the final width, which further justifies the assumption that only the variation in the region close to p_c is important for the properties of the pinned interface.

3. RESULTS

3.1 Interface Growth in (1 + 1) Dimensions

The presence of the gradient in the density of blocked cells changes the width (see Fig. 4) of the pinned interface and its scaling form. Our results for the pinned interface show that the saturated width behaves as $w \sim \ell^\alpha$ for $\ell \ll \xi$ and as $w \sim (\nabla p)^\gamma$ for $\ell \gg \xi$, where $\xi = \xi(\nabla p)$ is the correlation length. We find:

$$\alpha = 0.63 \pm 0.01\,, \quad \gamma = 0.52 \pm 0.01\,. \tag{9}$$

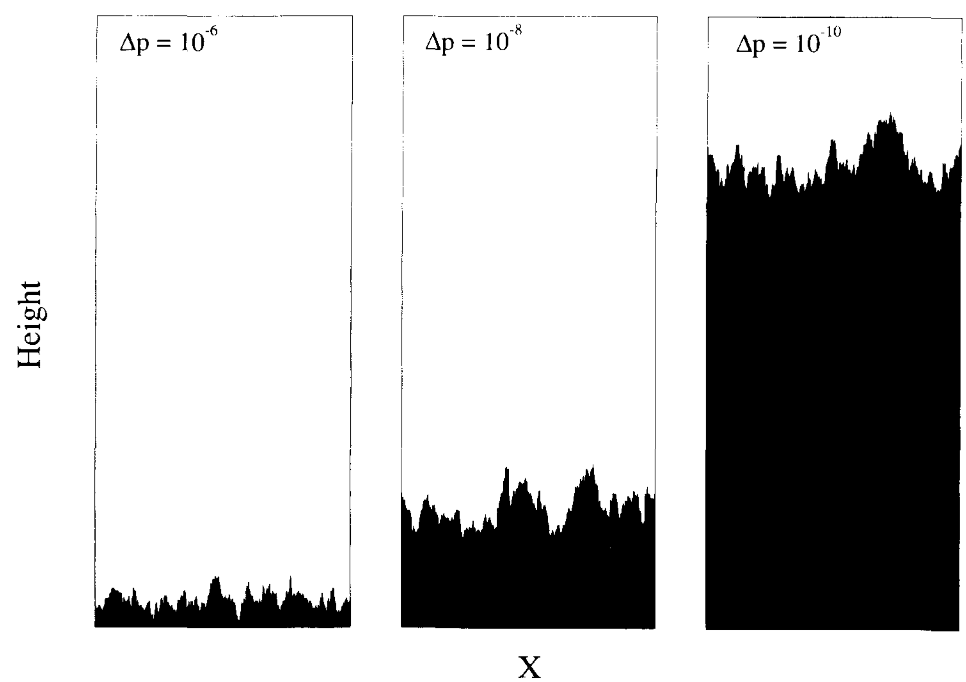

Fig. 4 Contour of completely pinned interfaces for three values of gradient: $\nabla p = 2^{-6}$, 2^{-8} and 2^{-10}. We can observe the increase in the final heights of the interface with the decrease of the gradient. For the widths, we can see an increase from the larger gradient to the other values but not between the smaller values. The reason for this lies in the fact that the system size is smaller than the correlation length ξ.

Moreover, we find our data to scale as:

$$w(\ell, \nabla p) \sim \ell^{\alpha} f\left(\frac{\ell}{(\nabla p)^{-\gamma/\alpha}}\right), \tag{10}$$

where the function $f(x)$ satisfies $f(x \ll 1) \sim const$ and $f(x \gg 1) \sim x^{-\alpha}$.

The value of α can be understood from the mapping to directed percolation, since the conditions for a complete pinning of the interface do not change from the model of Ref. 7. The presence of the gradient, which introduces a new length scale in the direction normal to the interface, is not important for $\ell \ll \xi$. On the other hand, for $\ell \gg \xi$, this second length scale will cause an early breakdown of the scaling behavior. By an argument similar to the one developed in Ref. 16 (see also Refs. 19 and 20) we can relate the exponent γ to $\nu_\perp$,

$$\gamma = \frac{\nu_\perp}{1 + \nu_\perp}. \tag{11}$$

This relation predicts $\gamma \cong 0.523$ in $(1 + 1)$ dimension, in very good agreement with our result [Eq. (9)].

3.2 Interface Growth in $(2 + 1)$ Dimensions

The $(2 + 1)$ dimensional model can be mapped to directed surfaces,[8,9,21] a percolation problem that has not been thoroughly investigated. A study along the lines described above for $(1 + 1)$ dimensions allows us to estimate the values of the critical exponents and critical probability for that problem.

From our data we find:

$$\alpha = 0.43 \pm 0.03\,, \quad \gamma = 0.32 \pm 0.01\,. \tag{12}$$

From these results, we calculate the values of the transverse and longitudinal correlation exponents for the directed surfaces problem,

$$\nu_\perp = 0.47 \pm 0.02\,, \quad \nu_{||} = 1.1 \pm 0.1\,. \tag{13}$$

Note that these results represent more accurate values for the exponents than those found earlier.[8]

Table 1 Our Results for the Critical Probability and Exponents

	$(1+1)$ dimensions	$(2+1)$ dimensions
p_c	0.47 ± 0.03	0.75 ± 0.03
α	0.63 ± 0.01	0.43 ± 0.03
γ	0.52 ± 0.01	0.32 ± 0.01
$\nu_\perp$	1.09 ± 0.01	0.47 ± 0.02
$\nu_{\|\|}$	1.73 ± 0.02	1.1 ± 0.1

4. CONCLUSIONS

We used the growth rule proposed in Ref. 7 to simulate the motion of the wet interface. The difference is that while in Ref. 7 the pinning sites were randomly distributed with a constant probability, p, in our model this probability changes with height according to Eq. (8). While the model of Ref. 7 needs to be tuned to reach the critical state at $p = p_c$, in the gradient model the system evolves naturally to the pinned critical state. However, since the gradient introduces a correlation length in the transverse direction we would hesitate to refer to this *self-tuning* ability of the model as self-organized criticality.[22]

An important conclusion from this work is the ability of our model to reproduce the important features of imbibition experiments. We observe both in simulations and experiments a dependence of the final height of the interface on the value of the gradient. We also find that a gradient in the density of pinning sites leads to the introduction of a new correlation length, in qualitative agreement with experiments. We verified the dependence of that correlation length on the gradient and its effect on the scaling properties of the interface. For the saturated pinned interface, we find good scaling of our data both in $(1+1)$ and $(2+1)$ dimensions. This allow us to estimate, with good accuracy, the critical exponents and the critical probability both for directed percolation and directed surfaces.

There are still a number of open questions related to the present model and the general problem of nonequilibrium roughening with quenched noise. A first question is the behavior near the pinning threshold. As we approach p_c, the interface velocity approachs zero as:

$$v \sim \left(1 - \frac{p}{p_c}\right)^{\zeta}. \tag{14}$$

Numerical integration of the KPZ equation with quenched noise suggests an exponent $\zeta = 0.64 \pm 0.08$.[23] In the presence of a gradient in the pinning force, this behavior is no longer valid. Recently, related models *without threshold* have been introduced,[24,25] which generate an interface with the same roughness exponent as in our model. It is very interesting that these models result in a nontrivial temporal multi-affinity,[26] while no spatial multi-affinity[27] has been observed. The question of the existence of this nontrivial multi-scaling in our model is still open.

It is important to understand the discrepancy between the different reported results of the roughness exponent. In the present model, since the pinning is due to the directed percolation cluster determined by the disorder, the roughness exponent is $\alpha \cong 0.63$. However different results have been reported from numerical integration of the KPZ equation (see Ref. 4) with quenched noise, resulting in $\alpha = 0.71 \pm 0.08$.[23] Renormalization group analysis leads to $\alpha = 1$,[28] while related scaling arguments gave 3/4.[23] So far an understanding of these discrepancies is not available.

After this work was completed we received a theoretical analysis by Olami *et al.*[29] that studies the behavior of the $(1+1)$ dimension model at criticality ($p = p_c$) and relates all the scaling exponents to a single one, α.

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