Maximum angle of stability in wet and dry spherical granular media

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(Received 9 April 1997)

We demonstrate that stability criteria can be used to calculate the maximum angle of stability θ_m of a granular medium composed of spherical particles in three dimensions and circular disks in two dimensions. The predicted angles are in good agreement with the experimental results. Furthermore, we determine the dependence of θ_m on cohesive forces, applying the results to wet granular material by calculating the dependence of θ_m on the liquid content of the material. We have also studied wet granular media experimentally and find good agreement between the theory and our experimental results. [S1063-651X(97)50512-5]

PACS number(s): 83.70.Fn, 05.90.+m, 46.10.+z, 68.45.−v

Granular materials display a variety of behavior that distinguishes them from other forms of matter. Unlike solids, granular media conform to the shape of a container and will flow if the container is tilted sufficiently. Unlike liquids, however, a granular material is stable when its container is tilted slightly as long as the top surface is at a slope less than the angle of maximal stability θ_m. When the slope is increased above θ_m, grains begin to flow and an avalanche of particles occurs, the angle of the pile decreasing to the angle of repose θ_r. However, instead of uniform motion throughout the sample, all of the motion occurs in a relatively thin (10 grains) boundary layer at the surface [1].

Experimental measurements of the angle of repose [2–4] reveal that θ_r depends strongly on the shape and surface roughness of the grains. The typical measured value for θ_r is $\simeq 22°$ for smooth spheres, but θ_r can attain 64° for materials containing rough, irregular particles. Cohesion between grains can also dramatically change the physical properties of a granular material, including θ_r and θ_m [5]. Such cohesion is commonly caused by the presence of a liquid in the material that forms interstitial bridges resulting in attractive forces between grains.

While many experimental measurements of θ_r and θ_m have been made for different materials, few theoretical results are available regarding the numerical values of these angles. The most detailed theoretical predictions are provided by molecular dynamics studies [6–8], which have profoundly improved our understanding of θ_r, but have not provided a simple way to calculate θ_r or θ_m.

In this paper we demonstrate that stability criteria can be used to calculate θ_m of spherical particles in three dimensions (3D) and circular discs in two dimensions (2D). The predicted angles are in good agreement with the experimental results. Furthermore, we determine the dependence of θ_m on cohesive forces, applying the results to wet granular material for which we have determined the dependence of θ_m on liquid content. We have also studied wet granular media experimentally, and we find good agreement between the theory and our experimental results.

Stability criteria. The basic idea of our approach is best illustrated in 2D. Consider a randomly packed sandpile of disks with equal radii, as shown in Fig. 1. If we add one more particle to the pile on a randomly chosen local surface minimum (see Fig. 1), its stability will depend entirely on the configuration of the two supporting particles underneath it [9]. To quantify the stability criteria we define the local slope of the sandpile θ as being the tangent to the two supporting spheres. If θ is small, the newly added particle is stable, while if θ is larger than a critical value θ_c, it is unstable and will roll down, starting an avalanche on the surface. For disks with equal radii, simple geometrical considerations indicate $\theta_c = 30°$.

This argument can be generalized to 3D, but the geometry is more complicated: we have to study the arrangement of three spherical particles supporting a fourth sphere. To simplify the presentation we consider spheres with equal radii, but generalizing to an arbitrary size distribution is straightforward. Again, the local slope of the sandpile θ is defined as the angle between the tangent plane to the spheres and the horizontal plane (see Fig. 2). For $\theta = 0$ the top sphere is stable, being supported by the three base spheres. Increasing θ, the top sphere remains in equilibrium for $\theta < \theta_c(\phi)$, where $\theta_c(\phi)$ depends on the relative orientation of the base spheres, quantified by the angle ϕ (see Fig. 2).

Without any cohesive or frictional forces, the top sphere is stable only while the gravitational force vector points within the projection of the base triangle on the horizontal plane. This criterion gives the maximum angle of stability as a function of ϕ,

$$\theta_c(\phi) = \arctan \frac{1}{2\sqrt{2}\,\cos(\pi/3 - \phi)}. \tag{1}$$

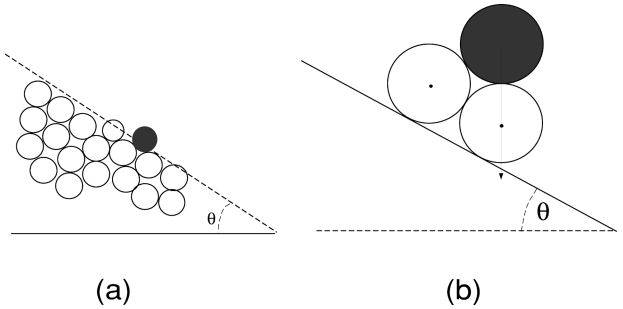

FIG. 1. (a) Schematic illustration of the surface of a two-dimensional sandpile and (b) the local surface configuration when a new particle (filled circle) is added to the pile. The stability of the newly added particle depends on the local slope of the surface.

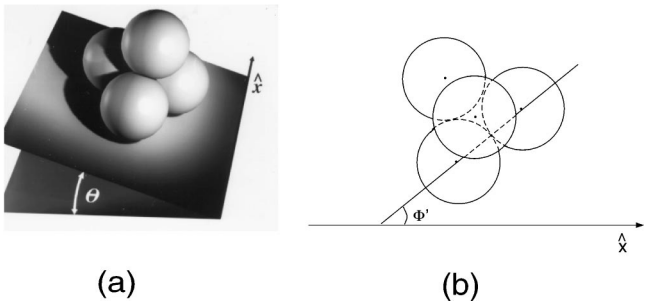

FIG. 2. (a) In three dimensions a newly added particle (top sphere) is supported by three surface particles. The local slope, denoted by θ, is the angle between the plane tangent to the three supporting sphere and the horizontal plane. The angle is increased by rotating the inclined plane around the axis $\hat{x}$. (b) Top view of the four spheres. The orientation of the triangle defined by the three supporting particles is characterized by the angle ϕ, related to the angle shown in the figure as $\phi = \pi/3 - \phi'$, where $0 < \phi < \pi/3$.

Since the coordination of the base triangle, described by ϕ, is random on a pile, one expects that the critical angle of a randomly packed pile will be given by the average $\theta_c(\phi)$, i.e., $\theta_c = (3/\pi)\int_0^{\pi/3}\theta_c(\phi)d\phi$. After performing this integral numerically we obtain $\theta_c = 23.4°$.

Beads used in actual experiments always have some microscopic surface roughness. If we consider that the particle, after loosing its static stability, is still immobile until it overcomes a static rolling friction force between itself and the supporting particle, θ_c will be increased by a correction proportional to $\arctan(\mu)$, where μ is the coefficient of rolling friction.

The question now arises as to whether we can associate θ_c with either θ_r or θ_m. Our calculated value of θ_c corresponds to the angle at which an ideal pile becomes unstable, where by ideal pile we mean that all its surface beads have the same local slope, and that the orientation of the base triangles, defined by ϕ, is completely random and uniformly distributed. In a real sandpile, there is always some surface roughness, and the local slopes vary along the surface. Consequently, the local surface configuration is defined by the total slope, $\theta_T = \theta_{\rm av} + \delta\theta$, where $\theta_{\rm av}$ is the average slope of the pile, and $\delta\theta$ is the local variation of the slope generated by the surface roughness. In this picture θ_m is the angle at which we have the first local configuration satisfying $\theta_m + \delta\theta > \theta_c(\phi)$. For a finite pile, where each local configuration has a certain probability to appear, θ_m is random, and its variance and average value is defined by the randomness in $\delta\theta$ (controlled by the surface roughness) and in ϕ. For a stationary pile one expects some nontrivial coupled distribution $P(\delta\theta, \phi)$, which could be determined experimentally. This distribution just *before* an avalanche provides θ_m, and right *after* the avalanche provides θ_r. In calculating θ_c by averaging over all values of ϕ, we have assumed that (i) ϕ and $\delta\theta$ are decoupled, (ii) ϕ is uniformly distributed between 0 and $\pi/3$, and (iii) the pile is not rough, i.e., the distribution of $\delta\theta$ is a δ function. These assumptions are equivalent to neglecting fluctuations in the pile, and thus θ_c represents a *mean field* approximation for θ_m. The distribution $P(\delta\theta, \phi)$ could be determined either experimentally or using molecular-dynamics simulations, and if an empirical fit for this function becomes available, that could be used to calculate corrections to the mean field result θ_c, and explicitly evaluate both θ_r and θ_m.

TABLE I. Comparison between the theoretical and experimental results for smooth noncohesive particles.

D	Material	θ_r	θ_m	θ_c (theory)	Reference
2	Disks	$24 \pm 1°$	33°	30°	10
2	Plastic disks	30°		30°	11
3	Glass beads	$22 \pm 2°$		23.4°	18
3	Glass beads	$23 \pm 2°$		23.4°	19
3	Glass beads	$22 \pm 5°$		23.4°	19
3	Polystyrene beads	$22 \pm 1°$		23.4°	this work
3	Glass beads	26°	28.6°	23.4°	3

The obtained mean field predictions for θ_c agree rather well, however, with experimental measurements of θ_r and θ_m, both for disks in 2D and spheres in 3D. The pertinent experiments are summarized in Table I. Reference [10] provides $\theta_m = 33°$ in 2D, only slightly larger than our prediction $\theta_m = 30°$. Similarly, the dynamical angle $\theta \simeq 30°$ given by the rotating drum experiment for low rotation speed in Ref. [11] coincides with our result. As Table I indicates, in 3D, with one exception, $\theta_r \simeq 22°$, indeed only slightly smaller than our prediction of $\theta_c = 23.4°$. This difference between θ_c and the experimental values is consistent with the experimentally observed difference between θ_r and θ_m for spherical particles [12].

The slightly higher value reported by Ref. [3] can be attributed to friction. However, we can identify another mechanism that could increase the experimental values of θ_m: we have assumed so far that in 3D the three supporting particles are close packed, i.e., they touch each other. Upon a detailed inspection of the surface of a granular pile one can observe that this rarely happens: typically, only two of the supporting particles touch each other, while the third particle of the base triangle is further away, having only one or no contact at all with the other two supporting spheres, thus increasing θ_c. Such corrections *can be included* into our calculation by studying the statistics of the base configurations in real sandpile, and using these statistics to recalculate the θ_c. Finally, experiments with *nonspherical* particles give typically much larger values for θ_m and θ_r. This effect is expected, since, for example, stacked pentagons have configurations that will stay stable for much larger slopes, thereby increasing θ_m. Indeed, experiments find that $\theta_r = 37°$ and $\theta_m = 45°$ for pentagons in 2D [10], while $\theta_r = 33 \pm 2°$ for real sand in 3D, constituted by nonspherical particles [2].

Cohesive forces. In the following we extend our results to the case when cohesive forces act between the particles [13]. We have performed experiments to investigate the transition from dry to wetted granular media by measuring the angle of repose of spherical polystyrene beads (diameter 0.8 ± 0.2 mm) by the draining crater method with aperture diameter of 2.5 cm [2]. These measurements were also conducted after small quantities of oil were added to the spheres with liquid content $t_{\rm liq}$ varying from zero to a maximum average coating thickness of 28 nm (four orders of magnitude less than the radius of the beads). Details of these ex-

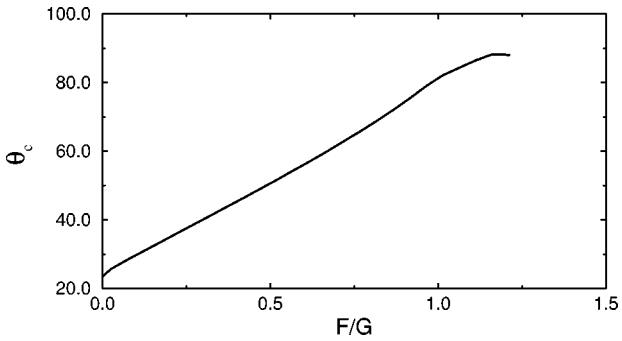

FIG. 3. The dependence of θ_c (in degrees) on the cohesive force as predicted by Eq. (2). Here F/G is the fraction between the cohesive force and the weight of the top sphere.

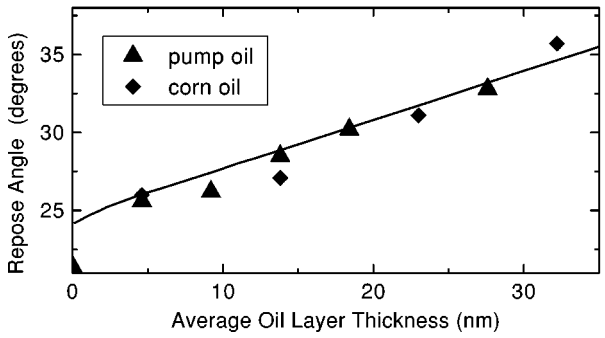

FIG. 4. Experimental measurement of θ_r as a function of oil content for two types of oil. Note the dramatic increase in θ_r with only a nanometer scale coating of oil. The solid curve is a fit of our model to the data where the only fitting parameter is the volume of oil in the intergrain bridges.

periments will be published elsewhere [15]. When no oil was added to the spheres, we found $\theta_r(t_{\mathrm{liq}=0}) \simeq 22°$, in good agreement with previous measurements (see Table I). As Fig. 4 shows, θ_r increased very rapidly with t_{liq}. The dependence of θ_r on t_{liq} is nearly linear up to the regime where clumping occurs, preventing an accurate determination of θ_r [15].

In the presence of cohesive forces acting between the beads, there are several forces acting on the top sphere: its weight, normal repulsive and cohesive forces at each contact with another sphere, and frictional forces. The maximal angle of stability is given by the condition of force equilibrium, and it is the solution of the equation

$$\frac{G}{F} = \frac{\sin\beta_2}{\sin(\gamma_1+\gamma_2)}\left(\frac{\sin\gamma_2}{\tan(\beta_1-f)} + \frac{\sin\gamma_1}{\tan(\beta_3-f)}\right) - \cos\beta_2, \tag{2}$$

where

$$\cos\beta_i = \frac{\sqrt{3}}{3}[\sqrt{2}\cos\theta - \sin(\phi+\alpha_i)\sin\theta], \tag{3}$$

$$\cos\gamma_1 = \frac{\sin^2\beta_1 + \sin^2\beta_2 + \sin^2\theta\sin^2\phi}{2\sin\beta_1\sin\beta_2}, \tag{4}$$

$$\cos\gamma_2 = \frac{\sin^2\beta_2 + \sin^2\beta_3 + \cos^2\left(\theta - \frac{\pi}{6}\right)\sin^2\phi}{2\sin\beta_2\sin\beta_3}, \tag{5}$$

with $i=1,2,3$, $\alpha_1 = -\pi/2$, $\alpha_2 = \pi/6$, and $\alpha_3 = \pi/3$ (see Fig. 3). Here G is the weight of one grain, F is the cohesive force, and f is the friction angle [$f = \arctan(\mu)$].

The expression (2) can be used to calculate θ_c in the presence of cohesive forces, and thus to describe quantitatively the transition from dry to wet granular media. But for this we first need to calculate the magnitude of the liquid-induced cohesive force F, and its dependence on the thickness of the liquid layer t_{liq}. The force between two particles connected by a liquid bridge,

$$F = 2\pi r\sigma\cos\psi - \pi r^2\Delta p, \tag{6}$$

consists of a surface tension term acting at the wetting perimeter $2\pi r$ and a term arising from the capillary pressure $\Delta p = \sigma[(1/\rho_2) - 1/\rho_1]$ in the liquid [see Fig. 5(a)]. Here σ is the surface free energy of the liquid-vapor interface, ρ_1 and ρ_2 are the radii of curvature of the liquid bridge. The angle ψ is related to the wetting angle, ψ_w, by the relation $\psi = (\pi/2) - \psi_w - \arcsin(r/R)$, where R is the radius of curva-

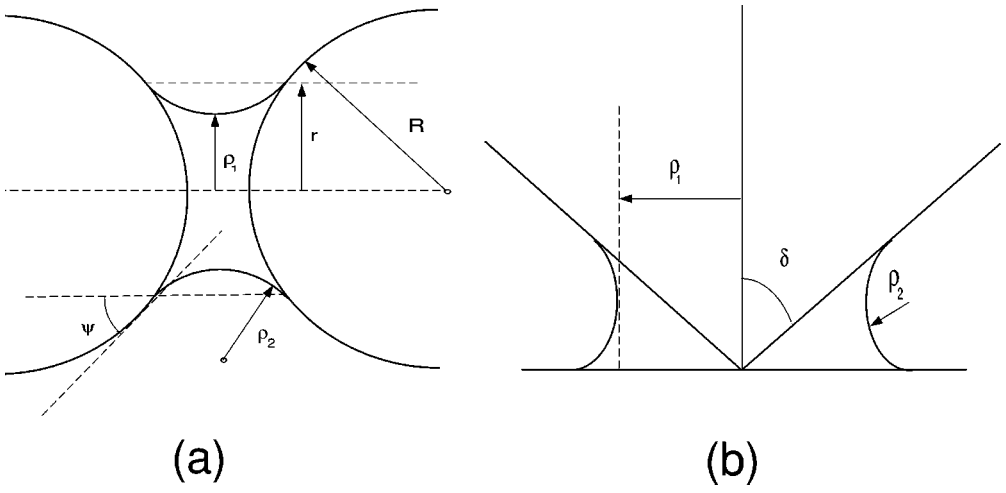

FIG. 5. A schematic of the liquid bridge at the particle-particle contact region: (a) sphere-sphere contact, assuming ideal, smooth spheres, (b) cone-plane contact, for grains that are microscopically rough.

ture of the solid surface. The strength of the cohesive force depends on the geometry of the contact between the two spheres. As a first approximation, one is tempted to consider an ideal sphere-sphere contact [Fig. 5(a)]. However, in the case where the spheres are in direct physical contact, F will decrease as the liquid content in the bridges increases [16]. Since one expects [based on Eq. (2)] that with increasing F the angle θ_c also increases, ideal sphere-sphere contact would imply that increasing the liquid content decreases θ_c, and consequently the repose angle [16], contradicting the experimental result of Fig. 4. This apparent paradox is resolved by incorporating the surface roughness of the individual particles, which prevents ideal sphere-sphere contact. Indeed, the surface of real granular particles is rough, and the contact is better approximated as a cone-plane type [see Fig. 5(b)]. The beads used in the experiment have a surface roughness $\approx \mu m$, much larger than the average oil thickness on the beads. Thus, at the length-scale set by t_{liq} the surface of the beads is very rough, supporting the applicability of the cone-plane contact.

In the cone-plane case the adhesive force increases monotonically with the liquid content, as $F = g(\delta) V_{\mathrm{bridge}}^{1/3}$, where the angle δ is defined in Fig. 5(b), V_{bridge} is the volume of the liquid bridge and the function g depends only on δ [16]. We have taken the half-angles of the cones in the interval $40° < \delta < 60°$. Since the function $g(\delta)$ does not vary strongly in this interval, we used its average value in our calculations.

To fit to the experimental results, we must calculate the volume of a liquid bridge, which in turn will give the cohesive force F. We could estimate V_{bridge} assuming that the liquid uniformly coats the surface of the beads; however, since the surface of the beads is rough, considerable quantities of liquid are in the ''valleys'' of the bead's landscape. To proceed, we assume that the volume of the liquid bridges V_{bridge} is an unknown parameter, and we calculate it fitting the theoretical curve to the experimental results [17]. Indeed, as is shown in Fig. 4, using $V_{\mathrm{bridge}} = 3 \times 10^{-17}$ m^3 for the maximum t_{liq}, the fit to the experiment is excellent, reproducing not only the long asymptotic linear part, but also the deviation from linearity for small t_{liq}.

It is important to note that this is a *one parameter* fit, that fixes only one point (that with maximum t_{liq}) to the experimental results, the rest of the curve being completely determined by this single parameter. Our fitted value of V_{bridge} should be compared with the average volume of liquid on a bead's surface of $V \simeq 5.5 \times 10^{-14}$ m^3, indicating that the actual volume of a liquid bridge is a factor of 1000 smaller than the average liquid content on a bead, i.e., a high fraction of the liquid is ''passive,'' not contributing to the cohesive forces between the beads. This is expected since the contact regions between the beads are small relative to their total bead surface area.

In conclusion, by considering the stability of the surface of a granular pile, we have presented the first analytical calculation of the critical angle in a granular medium and identified its relation to the repose angle and the maximum angle of stability. We have included the effects of cohesive forces, and we find good agreement with experimental data for both dry and wet material.

R. A. and A.-L. B. were partially supported by the NSF-CAREER Grant No. NSF/DMR 97-01998.

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