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The physics of sand castles: maximum angle of stability in wet and dry granular media

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Abstract

We demonstrate that stability criteria can be used to calculate the maximum angle of stability, θ_m, of a granular medium composed of spherical particles in three dimensions and circular discs in two dimensions. We apply the results to wet granular material by calculating the dependence of θ_m on the liquid content of the material. The results are in good agreement with our experimental data.

Keywords: Sand castles; Granular medium; Maximum angle

1. Introduction

Any child playing on the beach knows that the addition of a liquid (i.e. water) to a granular medium such as sand can dramatically change the granular properties – allowing the construction of sharp-featured sand castles which would be unstable in dry sand. The fact that the addition of liquid should have such a dramatic effect can be understood from a microscopic perspective since dry grains interact only through repulsive contact forces while wet grains also attract each other with an adhesive force associated with the liquid bridges between grains. Therefore, the addition of liquid changes the system from one with a hard-core repulsive interaction to one with both a repulsive and an attractive intergrain interaction. Many of the physical phenomena associated with a granular medium, such as the angle of repose, the formation of patterns in a vibrated granular layers, the segregation of particles of different size, the dynamics of avalanches on the surface, etc. will likely be affected by the presence of liquid. In this paper we investigate experimentally and theoretically the angle of repose of dry and wet granular media. We also discuss another consequence of wetness in granular matter: liquid-induced clumping.

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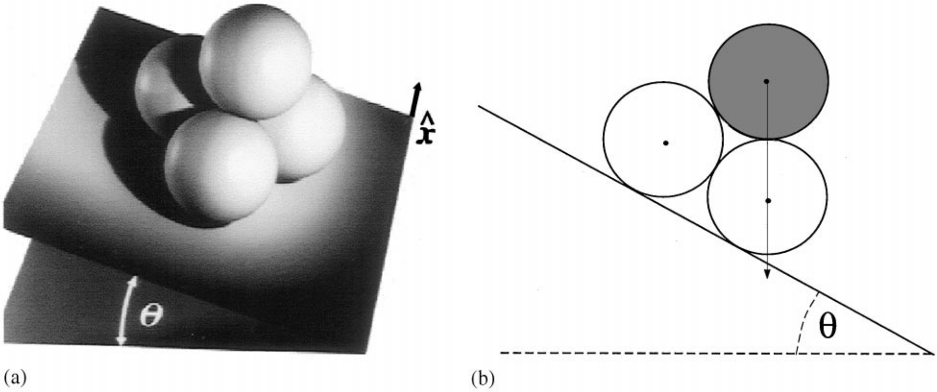


Fig. 1. (a) Schematic illustration of the local surface configuration when a new particle (filled circle) is added to the pile. (b) In three dimensions a newly added particle (top sphere) is supported by three surface particles. The local slope, denoted by θ, is the angle between the plane tangent to the three supporting sphere and the horizontal plane.

2. Angle of repose in dry granular media

A granular material can be stable when its container is tilted slightly as long as the top surface is at a slope less than the angle of maximal stability, θ_m. When the slope is increased above θ_m, grains begin to flow and an avalanche of particles occurs, the angle of the pile decreasing to the angle of repose, θ_r. [1] While many experimental measurements of θ_r and θ_m have been made for different materials, few theoretical results are available regarding the numerical values of these angles. The most detailed theoretical predictions are provided by molecular dynamics studies [2–4], which have profoundly improved our understanding of θ_r, but have not provided a simple way to calculate θ_r or θ_m.

2.1. Stability criteria

The basic idea of our approach is best illustrated in 2D. Consider a randomly packed sandpile of discs with equal radii. If we add one more particle to the pile on a randomly chosen local surface minimum (see Fig. 1), its stability will depend entirely on the configuration of the two supporting particles underneath it [5]. To quantify the stability criteria we define the local slope of the sandpile, θ, as being the tangent to the two supporting spheres. If θ is small, the newly added particle is stable, while if θ is larger than a critical value θ_c, it is unstable and will roll down, starting an avalanche on the surface. For discs with equal radii, simple geometrical considerations indicate $\theta_c = 30^\circ$. This argument can be generalized to 3D, but the geometry is more complicated: we have to study the arrangement of three spherical particles supporting a fourth sphere. Without any cohesive or frictional forces, the top sphere is stable only while the gravitational force vector points within the projection of the base triangle

Table 1
Comparison between the theoretical and experimental results for smooth non-cohesive particles

D	Material	θ_r	θ_m	θ_c (theory)	Ref.
2	Discs	$24 \pm 1^\circ$	33°	30°	[6]
2	Plastic discs	30°	–	30°	[1]
3	Glass spheres	$22 \pm 2^\circ$	–	23.4°	[7]
3	Glass spheres	$23 \pm 2^\circ$	–	23.4°	[8]
3	Glass spheres	$22 \pm 5^\circ$	–	23.4°	[8]
3	Polystyrene spheres	$22 \pm 1^\circ$	–	23.4°	this work
3	Glass beads	26°	28.6°	23.4°	[9]

[1]Data taken from Fig. 2 of Ref. [3], after Lebec.

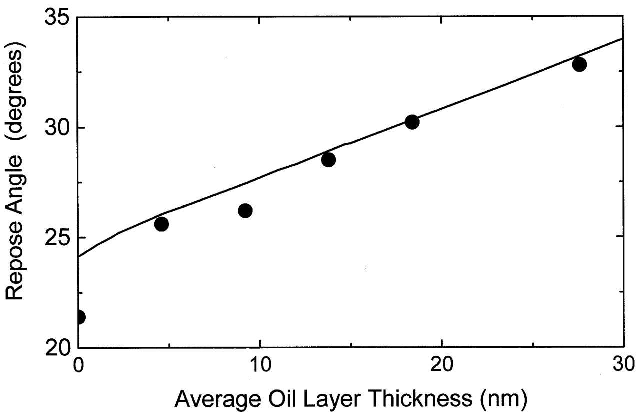


Fig. 2. Angle of repose as a function of oil content for two types of oil. The solid curve is a fit of our model to the data where the only fitting parameter is the volume of oil in the inter-grain bridges.

on the horizontal plane (Fig. 1). This criterion gives the maximum angle of stability as $\theta_c = 23.4^\circ$. As Table 1 indicates, these predictions for θ_c agree rather well with experimental measurements of θ_r and θ_m both for discs in 2D and spheres in 3D.

3. Angle of repose for wet granular media

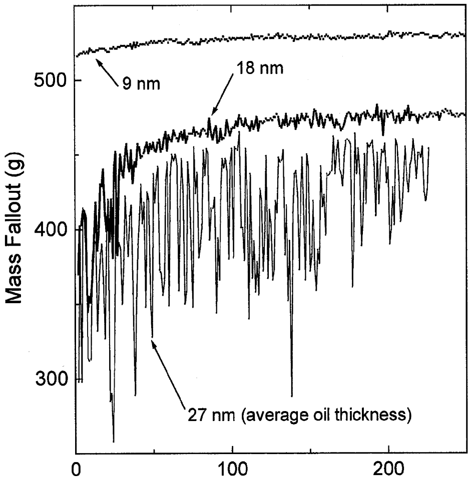
In the following we extend our results to the case when cohesive forces act between the particles. We have performed experiments to investigate the transition from dry to wet granular media by measuring the angle of repose of spherical polystyrene beads (diameter 0.8 ± 0.2 mm) by the draining crater method [9]. These measurements were also conducted after small quantities of oil were added to the spheres with liquid content t_{liq} [10,11]. When no oil was added to the spheres, we found $\theta_r(t_{\text{liq}=0}) \simeq 22^\circ$, in good agreement with previous measurements (see Table 1). As Fig. 2 shows, θ_r increased very rapidly with t_{liq}. The dependence of θ_r on t_{liq} is nearly linear up to the regime where clumping occurs, preventing an accurate determination of θ_r.

In the presence of cohesive forces acting between the beads, there are several forces acting on the top sphere: its weight, normal repulsive and cohesive forces at each contact with another sphere, and frictional forces. The condition of force equilibrium can be used to calculate θ_c in the presence of cohesive forces, and thus to describe quantitatively the transition from dry to wet granular media. But for this we first need to calculate the magnitude of the liquid-induced cohesive force F, and its dependence on the thickness of the liquid layer t_{liq}. The strength of the cohesive force depends on the geometry of the contact between the two spheres [12]. As a first approximation, one is tempted to consider an ideal sphere–sphere contact. However, the surface roughness of the individual particles prevents ideal sphere–sphere contact. Indeed, the beads used in the experiment have a surface roughness $\approx 1\,\mu\mathrm{m}$, thus at the length-scale set by t_{liq} the surface of the beads is very rough, and the contact is better approximated as a cone-plane type. In the cone-plane case the adhesive force increases monotonically with the liquid content, as $F = g(\delta) \cdot V_{\mathrm{bridge}}^{1/3}$, where δ is the half-angle of the cone, V_{bridge} is the volume of the liquid bridge and the function g depends only on δ [15]. To fit the experimental results, we must calculate the volume of a liquid bridge, which in turn will give the cohesive force F. We assume that the volume of the liquid bridges, V_{bridge}, is an unknown parameter, and we calculate it fitting the theoretical curve to the experimental results. Indeed, as is shown in Fig. 2, using $V_{\mathrm{bridge}} = 3 \times 10^{-17}\,\mathrm{m}^3$ for the maximum t_{liq}, the fit to the experiment is excellent, reproducing the long asymptotic linear part. [1]

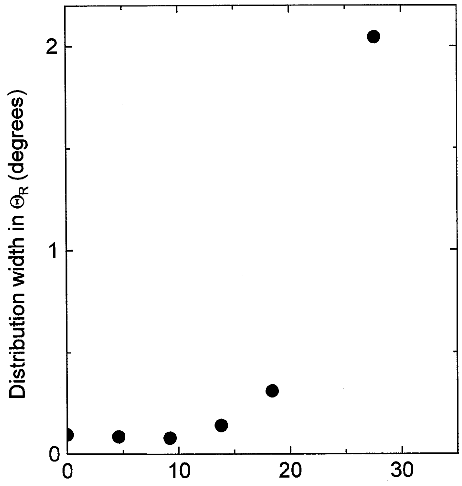
3.1. Clumping

One major distinguishing feature of wet granular media is the presence of correlations among the particles, which leads to the phenomenon of clumping. Experimental evidence for clumping is presented in Fig. 3. As liquid was added to the spheres, we observed the development of correlated particle clusters (clumps), whose size increased with liquid content. The presence of such clusters leads to an aperture dependence in θ_r for the largest values of t_{liq}. Our data also reflect the development of clustering as an increase in the width of the distribution of our measured values of θ_r with increasing t_{liq} (see Fig. 1b). Fig. 3a shows the mass fallout from the system as a function of the number of runs, for three different oil contents. One can see that as the t_{liq} increases, the fluctuations increase dramatically. This is quantified in Fig. 3b, where we show the

[1] Note that we used surface stability analysis to calculate the angle of repose of wet granular media. However, the interstitial liquid is expected to change the bulk stability of the granular matter as well. Calculations based on bulk stress analysis in wet granular materials predict [13,12] that the θ_r for wet and dry granular media coincide for infinite piles, while for finite systems the increase in θ_r depends on the size of the pile. We have performed experiments with different pile size, probing this effect. A detailed account of these experiment will be presented elsewhere [14].



(a)



(b)

Fig. 3. (a) The amount of material drained by the experimental apparatus as a function of the measurement number, illustrating the increase in the fluctuations with the increasing oil liquid content. The fallout mass is directly related to θ_r. (b) The width of the distribution of the angle of repose (θ_r) as a function of added pump oil. The symbols are the same as in Fig. 2.

width of the distribution. This width corresponds to the roughness of the crater surface. Note that the development of such clusters appears from these data to be rather sudden, suggesting a transition from a regime where the bulk properties are associated with the dynamics of individual grains to a regime where long-range correlations control the material behavior. Further evidence of the importance of correlated clusters is that our apparatus failed to drain entirely for larger $t_{\text{liq}} \geqslant 40$ nm when the size of the clusters approached the aperture size.

While at this point there is no definite answer regarding the origin of clumping, we do have a working hypothesis, that we plan to pursue in future work. If we consider a hypothetical clump in a wet granular medium, the stability against external forces of the clump will depend on the strength and the number of bonds within that region. Assuming uniform liquid coverage, clump formation can be the result of the local increase of the packing density of the granular matter. It is well known that for dry spheres random packing can be achieved at different packing densities, varying between 0.55 and 0.64 [1]. In dry randomly packed granular matter there are local density fluctuations between these two limits. A higher packing fraction will lead to a larger number of neighbors per particle. For wet particles, that translates into more attractive bonds per unit volume, leading to an increased stability of the higher density regions. It is a major challenge to understand the physical mechanism responsible for this clumping phenomenon, as well as to obtain the relationship between the clump size and the liquid content of the granular matter.

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